Power Spectral Density of ~~WSS~~ WSS R.P.s

Let $X(t)$ be WSS with autocorrelation function $R_X(\tau)$.

The power spectral density (PSD) of $X(t)$ is

$$S_X(f) = \mathcal{F}\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} \, d\tau$$

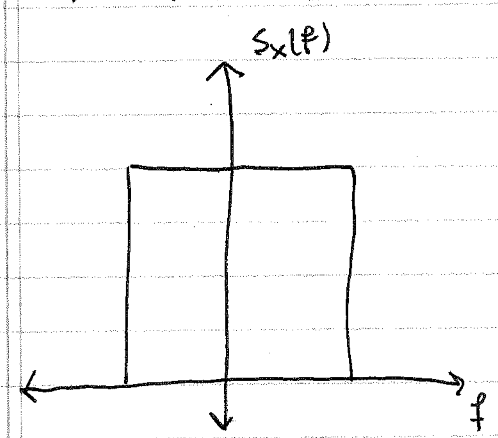
Properties of PSD

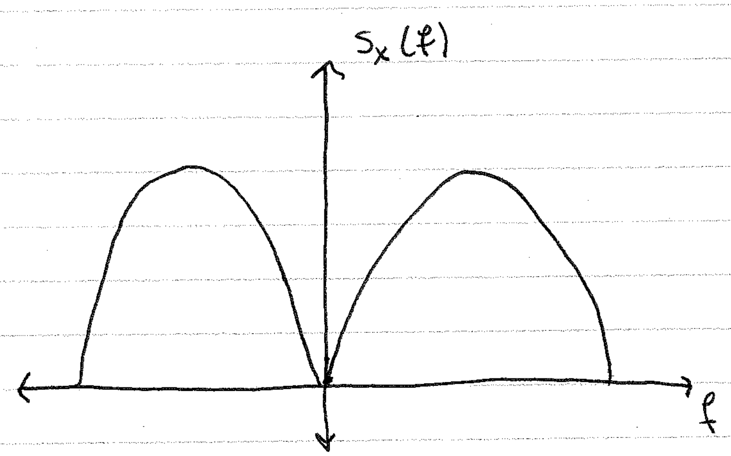
1) $E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f)\, df \geq 0$

2) $S_X(f) = S_X(-f)$ (PSD is even)

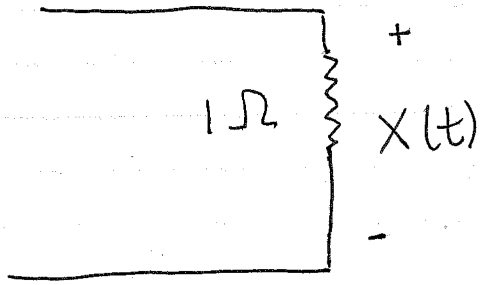
3) $S_X(f)$ is real-valued.

4) $S_X(f) \geq 0$





Interpretation



Let $X(t)$ be a (random) voltage across a $1\,\Omega$ resistor.

$E[X^2(t)]$ is the average power (in W) dissipated across the resistor. If $X(t)$ is WSS, can compute $E[X^2(t)]$ as

$$E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\,df.$$

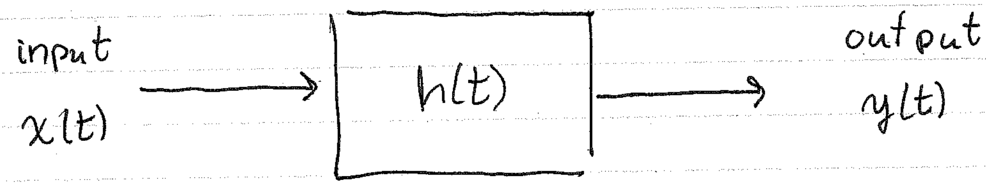
The PSD $S_X(f)$ gives the density of power of the random process $X(t)$ at a frequency f (in Hz).

Because of this, $S_X(f)$ is in units W/Hz, and can be used to find the average power along a bandwidth $B = [f_1, f_2]$ as

$$P_B = \int_{f_1}^{f_2} S_X(f)\,df$$

Response of LTI systems to Random signals

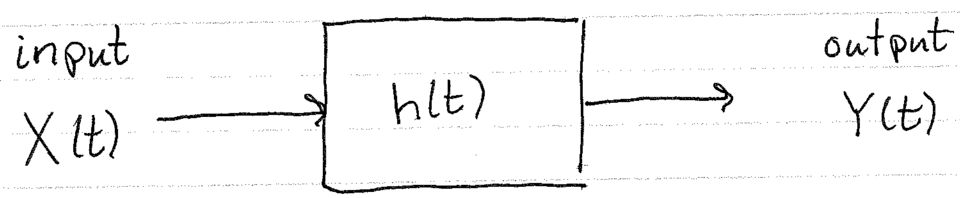
Consider an LTI system with impulse response $h(t)$.



The response of the system $y(t)$ to the input $x(t)$ can be found as $y(t) = x(t) * h(t)$

$$= \int_{-\infty}^{\infty} h(s)\, x(t-s)\, ds$$

What if the input is a ~~wss~~ WSS process $X(t)$?



What can be said about $Y(t)$?

Can show that $Y(t)$ is WSS.

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(s)\, X(t-s)\, ds\right]$$

$$= \int_{-\infty}^{\infty} h(s)\, E[X(t-s)]\, ds$$

$$= \int_{-\infty}^{\infty} h(s)\, \mu_X\, ds \quad , \text{ since } X(t) \text{ is WSS}$$

$$\Rightarrow \boxed{\begin{aligned} \mu_Y &= \mu_X \int_{-\infty}^{\infty} h(s)\,ds \\ &= \mu_X H(0) \end{aligned}}, \quad \text{where } H(f) = \mathcal{F}\{h(t)\}$$

is the transfer function of the system.

$\cancel{R_Y(t_1,}$

$$R_Y(t_1,t_2) = E[Y(t_1)\,Y(t_2)]$$

$$= E\left[\left(\int_{-\infty}^{\infty} h(s) X(t_1 - s)\,ds\right)\left(\int_{-\infty}^{\infty} h(r) X(t_2 - r)\,dr\right)\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)\,X(t_1-s)\,X(t_2-r)\,ds\,dr\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)\,E[X(t_1-s)\,X(t_2-r)]\,ds\,dr$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)\,R_X(t_2 - r - (t_1 - s))\,ds\,dr,$$

since $X(t)$ is WSS

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h(r)\,R_X(\underline{\underline{t_2 - t_1}} + s - r)\,ds\,dr$$

Letting $\tau = t_2 - t_1$ and $\tilde{h}(t) = h(-t)$

we have that

$$\boxed{R_Y(\tau) = R_X(\tau) * h(\tau) * \tilde{h}(\tau)}$$

Taking the Fourier transform of both sides

gives $S_Y(f)$ as

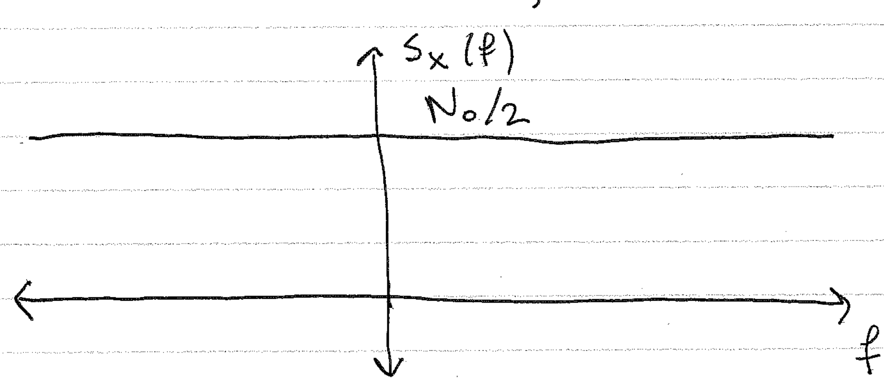
$$S_Y(f) = S_X(f) \cdot H(f) \cdot H^*(f)$$

$$\boxed{= |H(f)|^2 S_X(f)}$$

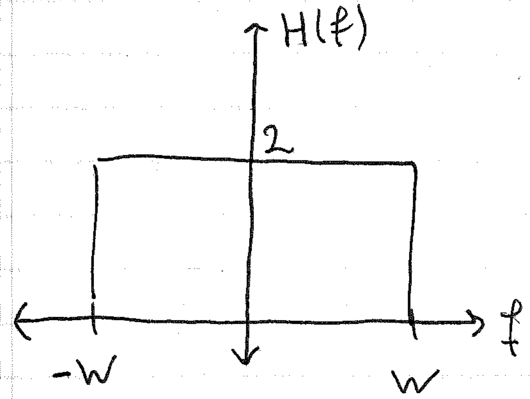
Therefore $Y(t)$ is WSS.

Ex: A process $X(t)$ is said to be white Gaussian noise if its power spectral density is given by

$$S_X(f) = N_0/2 \quad , \quad -\infty < f < \infty$$



Let $X(t)$ be the input into an LTI system with transfer function. $H(f)$.

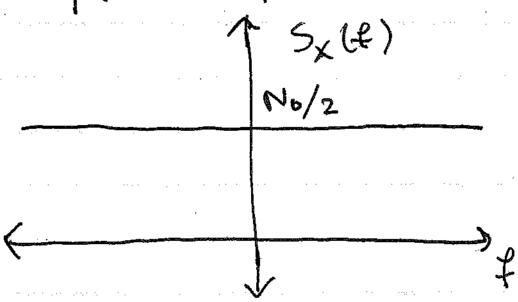


LPF Gain = 2

Bandwith 2W

Let $Y(t)$ be the output of the system.

$$S_Y(f) = |H(f)|^2 \, S_X(f)$$



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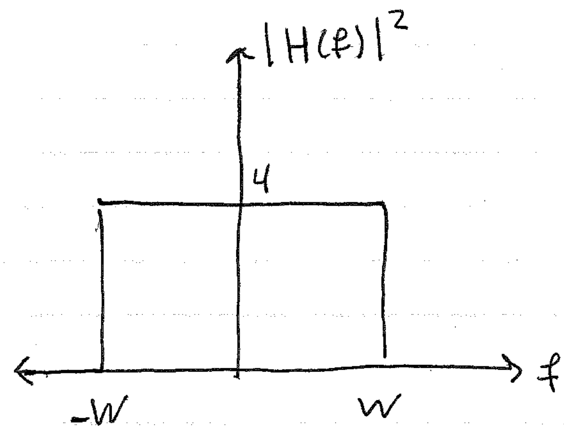

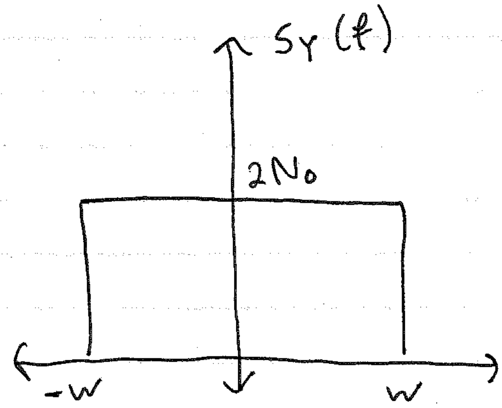

What is the average power of the output?

$$E[Y^2(t)] = \int_{-\infty}^{\infty} S_Y(f)\, df$$

$$= \int_{-W}^{W} 2N_0 \, df$$

$$= 4N_0W$$

If we want $E[Y^2(t)]$ (average power) to fall below a certain threshold, we can adjust W.