MA544 Qual Review Problems 2 Bridges Summer 08

1. Let $f \in BV(I)$. Show f is the difference of two monotone increasing functions.

2. Let $0 < p < 1$. Set $||\_||_* : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ by

$$||(x, y)||_* = |x|^p + |y|^p$$

and show that $||\_||_*$ induces a new metric. (Hint: Let $a \geq 0$, and consider $f(t) = a^p + t^p - (a + t)^p$). Is $||\_||_*$ a norm? What about $||(x, y)||_p = (|x|^p + |y|^p)^{\frac{1}{p}}$?

3. Continuity of measures from below: Let $(X, \Re, \mu)$ be a measure space. Show that if $E_n \in \Re$, $E_n \subseteq E_{n+1}$ then $\lim \mu(E_n) = \mu(\cup E_n)$.

   Continuity of measures from above (sometimes): Now let $E_n \supseteq E_{n+1}$. Suppose $\exists n$ such that $\mu(E_n) < \infty$ or that $\mu(\cap E_n) = \infty$. Show $\lim \mu(E_n) = \mu(\cap E_n)$. Find a counter example to show μ is not continuous from above; that is, $E_n \supseteq E_{n+1}$ and $\lim \mu(E_n) > \mu(\cap E_n)$.

4. Prove that f is lower semicontinuous if and only if $\{x \in \mathbb{R} : f(x) \leq \lambda\}$ is closed for every $\lambda \in \mathbb{R}$. Conclude that lower and upper semicontinuous functions are Lebesgue measurable.

5. Let f be a nonnegative funciton in $L^1(I_0)$ such that for each $n = 1, 2, \ldots$,

$$\int_0^1 f(x)^n dx = \int_0^1 f(x) dx.$$

   Show that $f(x) = \chi_E(x)$ almost everywhere for some measurable set $E \subset I_0$.

6. Suppose that $\{I_n\}$ is a sequence of disjoint nonempty open intervals contained in $[1/2, 1]$ such that $m(\bigcup_{n=1}^{\infty} I_n) = \frac{1}{2}$. If we write $I_n = (a_n, b_n)$, prove that

$$\sum_{n=1}^{\infty}\left(\frac{1}{a_n} - \frac{1}{b_n}\right) = 1.$$

7. Let C be a compact subset of $\mathbb{R}$ and assume that $f : C \to \mathbb{R}$ satisfies the following: for every $\alpha \in \mathbb{R}$, the set $\{x \in C : f(x) < \alpha\}$ is open in C. Show that there exists $x_0 \in C$ such that $f(x_0) = \sup_{x\in C} f(x)$.

8. (2-1) Let (X, d) be a metric space.

   (a) For (nonempty) $F \subset X$, let $f(x) = d(x, F) = \inf\{d(x, y) : y \in F\}$. Show that f is continuous.

   (b) Let K and F be nonempty subsets of X such that K is compact. Show that there is a p in K such that

   $$d(p, F) = \inf\{d(x, y) : x \in K, y \in F\}.$$

   (c) Asumme $K \subset U \subset X$ where K is compact and U is open. Show that there is an $r > 0$ such that $x \in K$ and $d(x, y) < r$ imply $y \in U$.

9. If $f \in C[a, b]$ and one of its derivatives (say D^+) is everywhere nonnegative on (a, b), then f is nondecreasing on $[a, b]$.

10. Decimal expansions in base b: If b is an integer larger than 1 and $0 < x < 1$, show that there exist integer coefficents, $c_k, 0 \le c_k < b$, such that

    $$x = \sum_{k=1}^{\infty} \frac{c_k}{b^k}.$$

    Show this expansion is uniue unless $x = \frac{c}{b^k}$, when there are two expansions.