6.2 In Section 5.5, the notion of a vector space of functions is introduced, and the *norm* and *inner product* are defined for functions in this space. The waveforms that arise in the binary communication system can then be illustrated by two-dimensional sketches as in Figures 6-7(a) and 6-7(b), where we use this geometric approach to illustrate the projections of the signals $s_0(t)$ and $s_1(t)$ onto the reference signal $r(t)$. The geometric approach can also be applied to obtain an intuitive description of the operation performed by a correlation receiver for binary signaling on the AWGN channel. Let the sum of the received signal and the noise be $Y(t)$, and consider the receiver shown in Figure 5-16(c). In this receiver, the two correlators are used to form the statistics

$$W_0 = \int_{-\infty}^{\infty} Y(t)\, s_0(t)\, \mathrm{d}t \quad \text{and} \quad W_1 = \int_{-\infty}^{\infty} Y(t)\, s_1(t)\, \mathrm{d}t.$$

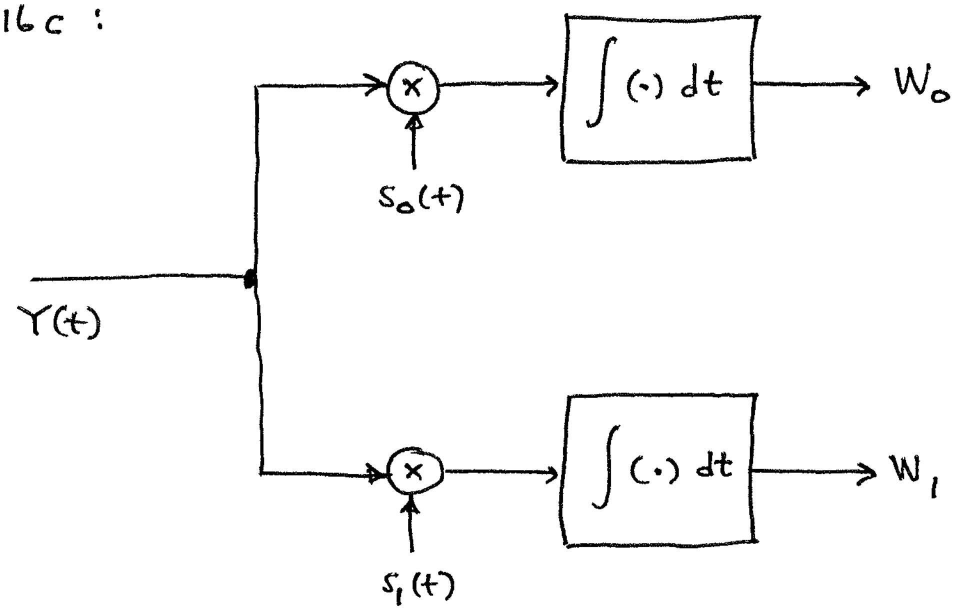
The optimum decision rule, as described in Section 5.5.4, is to decide that 0 was sent if $W_0 > W_1$, and to decide that 1 was sent if $W_1 > W_0$. (The event $W_1 = W_0$ has probability zero.) This rule decides in favor of the signal that has the larger correlation with the received waveform $Y(t)$. Written in vector notation, the decision rule is to decide that 0 was sent if $(Y, s_0) > (Y, s_1)$ and decide that 1 was sent if $(Y, s_1) > (Y, s_0)$. In parts **(a)** and **(b)**, assume that s_0 and s_1 have the same energy.

(a) Using the definition for the norm and inner product, show than an equivalent decision rule is to decide 0 was sent if $\|Y - s_0\| < \|Y - s_1\|$ and decide 1 was sent if $\|Y - s_0\| > \|Y - s_1\|$. Illustrate this fact geometrically. The *distance* between signals v_1 and v_2 is just $\|v_1 - v_2\|$, the norm of the difference between the two signals, so the conclusion is that the signal that has the largest correlation with the received waveform is the same as the signal that is closest to the received waveform.

(b) Apply the geometric viewpoint to the BPSK signal set defined by equations (6.3), and argue that an equivalent decision rule for this signal set is to determine the "phase" of the decision statistic and decide that 0 was sent if this phase is between $\varphi - (\pi/2)$ and $\varphi + (\pi/2)$ and 1 was sent if the phase is not in this range. In order to make this precise, how must the "phase" of the decision statistic be defined? *Hint*: The signals can be expressed as linear combinations of $\cos(\omega_c t)$ and $\sin(\omega_c t)$, and the operation of a correlation receiver matched to $\cos(\omega_c t + \varphi)$ can be expressed in terms of these two orthogonal functions. (What are the coefficients?) Assume that $\omega_c T$ is a multiple of 2π.

MBP 6.2

Fig. 5.16c :



Opt. decision rule is

$$(Y, s_0) > (Y, s_1) \implies \text{decide } H_0$$

$$(Y, s_0) < (Y, s_1) \implies \text{decide } H_1$$

(a)
Show equivalence to rule based on norm

$$\|Y - s_0\| < \|Y - s_1\| \implies \text{decide } H_0.$$

$$\|Y - s_0\| > \|Y - s_1\| \implies \text{decide } H_1$$

and illustrate geometrically.

Suffices to show:

$$(Y, s_0) > (Y, s_1) \iff \|Y - s_0\| < \|Y - s_1\|$$
$$\iff \|Y - s_0\|^2 < \|Y - s_1\|^2$$

$$\|Y - s_0\|^2 = (Y - s_0, Y - s_0)$$
$$= (Y, Y) - (Y, s_0) - (s_0, Y) + (s_0, s_0)$$
$$= \|Y\|^2 - 2(Y, s_0) + \|s_0\|^2$$

$$\|Y - s_1\|^2 = \|Y\|^2 - 2(Y, s_1) + \|s_0\|^2$$

$$\|Y - s_0\|^2 < \|Y - s_1\|^2$$

$$\Updownarrow$$

$$\cancel{\|Y\|^2} - 2(Y, s_0) + \cancel{\|s_0\|^2} < \cancel{\|Y\|^2} - 2(Y, s_1) + \cancel{\|s_1\|^2}$$

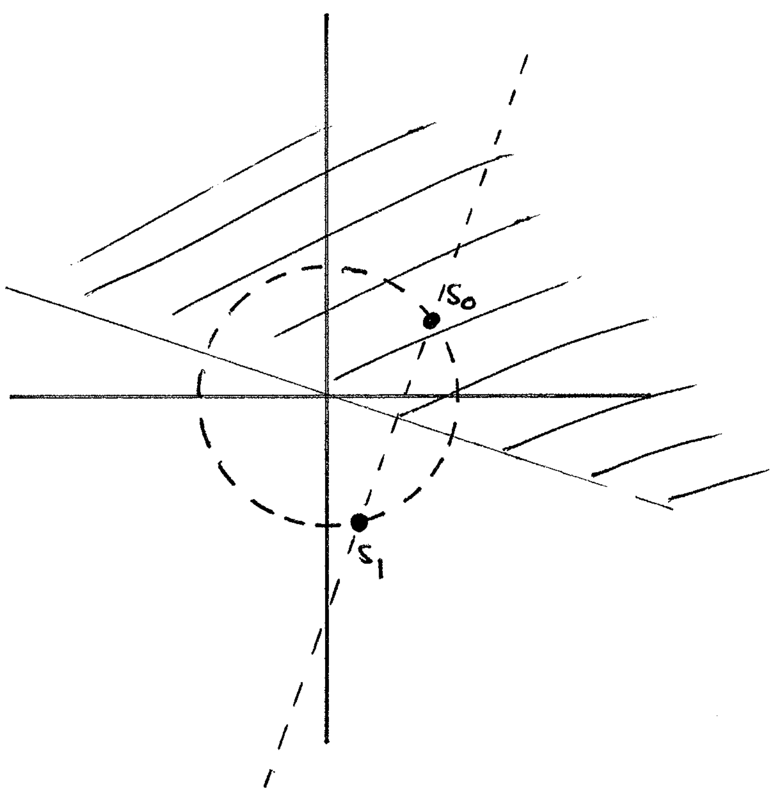
assuming signals have same energy

$$\Updownarrow$$

$$(Y, s_0) > (Y, s_1)$$

Geometric Illustration

Signals s_0 and s_1 of identical energy must be on a circle centered at origin:



6.3 This problem deals with the phase ambiguity in the squaring loop described in Section 6.2.2.

(a) Ignore the effects of noise at the input to the squaring loop. Show that if the output of the VCO is

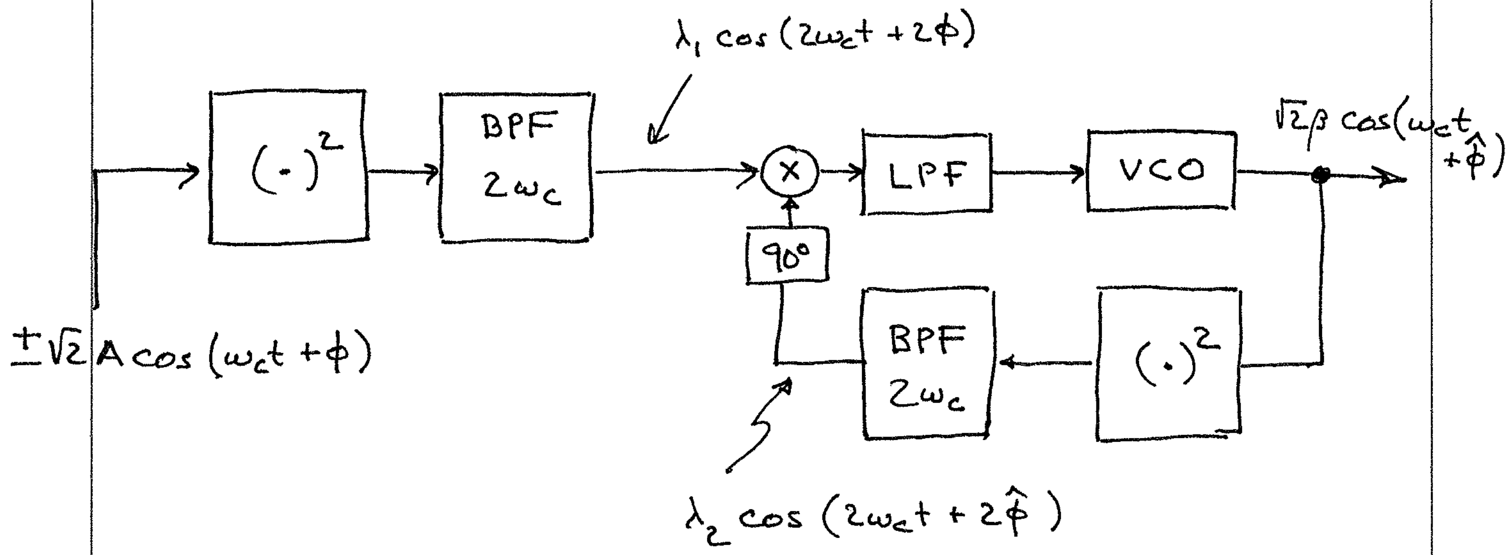
$$r(t) = \sqrt{2}\,\beta\cos(\omega_c t + \hat{\varphi})$$

and either $\hat{\varphi} = \varphi$ or $\hat{\varphi} = \varphi + \pi$, the loop is locked (i.e., the input to the VCO in the loop is zero). Thus, the loop "cannot tell" if there is a π-radian phase error in the output of the VCO.

(b) Suppose $r(t)$ is employed as a reference signal in a correlation receiver and the input to the receiver is $s_i(t) + X(t)$, as shown in Figure 6-6. Let Z be the decision statistic that results if $\hat{\varphi} = \varphi$. Show the decision statistic that results if $\hat{\varphi} = \varphi + \pi$ is $-Z$, so all decisions made by comparison with a zero threshold are reversed.

MBP 6.3

The squaring loop in question is that of Fig 6-9:



(a) Show that for either $\hat{\phi} = 0$ or $\hat{\phi} = \pi$ the loop is locked.

The inputs to the phase detector mixer are

$$\lambda_1 \cos(2\omega_c t + 2\phi)$$

and

$$\lambda_2 \sin(2\omega_c t + 2\hat{\phi})$$

Thus input to LPF is

$$\lambda_1 \lambda_2 \cos(2\omega_c t + 2\phi) \sin(2\omega_c t + 2\hat{\phi})$$

$$= \frac{\lambda_1 \lambda_2}{2} \left[-\sin 2(\phi - \hat{\phi}) + \sin(4\omega_c t + 2\phi + 2\hat{\phi}) \right]$$

Thus output of LPF (ie input to VCO) is

$$-\frac{\lambda_1 \lambda_2}{2} \sin 2(\phi - \hat{\phi})$$

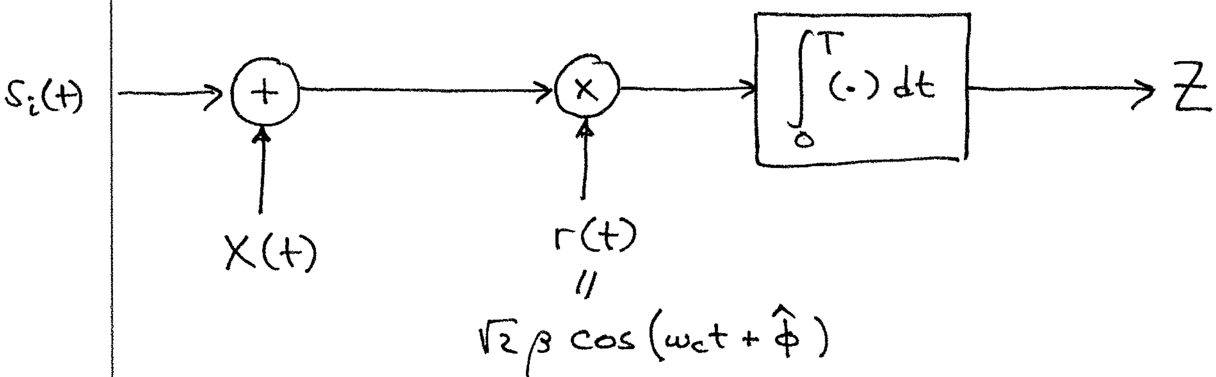
There needs to be another sign change assuming that VCO freq. increases with pos. input voltage.

Hence assume there is such sign change. Then output of LPF is

$$\frac{\lambda_1 \lambda_2}{2} \sin 2(\phi - \hat{\phi}) = \begin{cases} 0 & \text{if } \hat{\phi} = \phi \\ 0 & \text{if } \hat{\phi} = \phi + \pi \end{cases}$$

$\Longrightarrow$ In either case the VCO input voltage is zero, ie locked.

(b) Now this derived carrier is used in



In class we showed that the decision statistic in case of imperfect phase ref. was

$$Z = \left\{ \begin{array}{c} \mu_0 = \beta A T \cos(\phi - \hat{\phi}) \\ \text{or} \\ \mu_1 = -\beta A T \cos(\phi - \hat{\phi}) \end{array} \right\} + \left\{ \begin{array}{c} \text{a zero mean} \\ \text{Gaussian r.v.} \\ \text{of variance} \\ \beta^2 N_0 T/2 \end{array} \right\}$$

Clearly

(i) $\phi = \hat{\phi} \Longrightarrow \mu_0 = \beta A T, \ \mu_1 = -\beta A T$

(ii) $\phi = \hat{\phi} - \pi \Longrightarrow \mu_0 = -\beta A T, \ \mu_1 = +\beta A T$

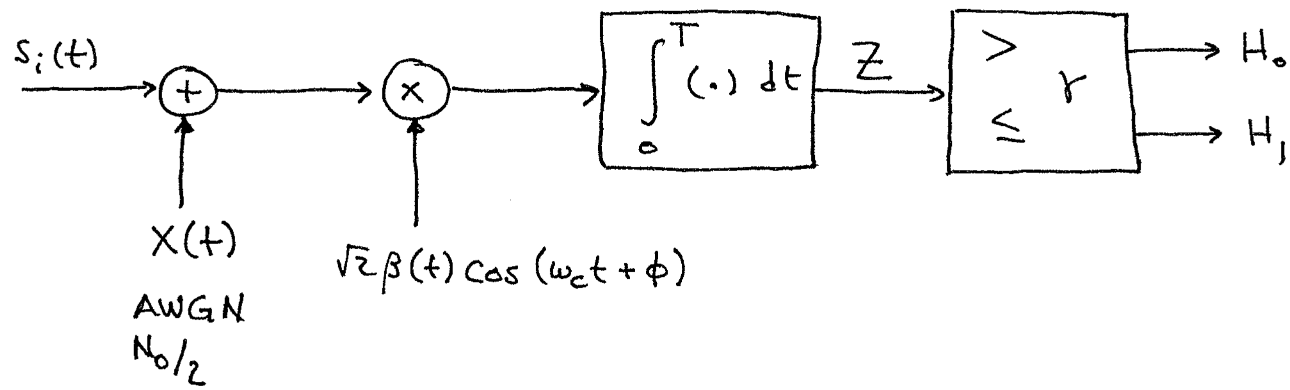
This shows the sign change expected.

6.7 Suppose the threshold for the receiver shown in Figure 6-12 is denoted by γ. The channel is an AWGN channel with spectral density $N_0/2$. Give expressions for $P_{e,0}$ and $P_{e,1}$ if the binary ASK signal set is

$$s_i(t) = \sqrt{2}\, A\, u_i\, p_T(t) \cos(\omega_c t + \varphi)$$

for $i = 0$ and $i = 1$ (i.e., $\beta(t) = p_T(t)$). Assume that $\omega_c \gg T^{-1}$ so that the double-frequency terms can be neglected. Express your answers in terms of the complementary Gaussian distribution function Q and the parameters u_0, u_1, γ, A, T, and N_0.

MBP 6.7



$$s_i(t) = u_i \sqrt{2} A\, p_T(t) \cos(\omega_c t + \phi) \qquad [\beta(t) = p_T(t)]$$

Say WLOG (without loss of generality) $u_0 > u_1$

Need to find statistical characterization of Z under the two hypotheses.

$$Z = u_i \int_0^T 2A \cos^2(\omega_c t + \phi)\, dt + \int_0^T X(t) \sqrt{2} \cos(\omega_c t + \phi)\, dt$$

$$= u_i A T + X$$

a zero mean, Gaussian r.v. with Variance $N_0 T/2$

$$\therefore \quad Z \sim \begin{cases} N(u_0 AT, N_0T/2) & \text{under } H_0 \\ N(u_1 AT, N_0T/2) & \text{" } H_1 \end{cases} \quad (u_0 > u_1)$$

$$P_{e,0} = P(Z \leq \gamma \mid H_0)$$

$$= P\left(\frac{Z - u_0 AT}{\sqrt{N_0T/2}} \leq \frac{\gamma - u_0 AT}{\sqrt{N_0T/2}} \,\middle|\, H_0 \right)$$

$$= \Phi\left(\frac{\gamma - u_0 AT}{\sqrt{N_0T/2}} \right) = Q\left(\frac{u_0 AT - \gamma}{\sqrt{N_0T/2}} \right)$$

$$P_{e,1} = P(Z > \gamma \mid H_1)$$

$$= P\left(\frac{Z - u_1 AT}{\sqrt{N_0 T/2}} > \frac{\gamma - u_1 AT}{\sqrt{N_0 T/2}} \,\middle|\, H_1 \right)$$

$$= Q\left(\frac{\gamma - u_1 AT}{\sqrt{N_0 T/2}} \right).$$

6.10 Suppose that the receiver of Figure 6-14 is used for the M-ASK signal set of Exercise 6-2, and the maximum-likelihood decision regions are employed. Show that

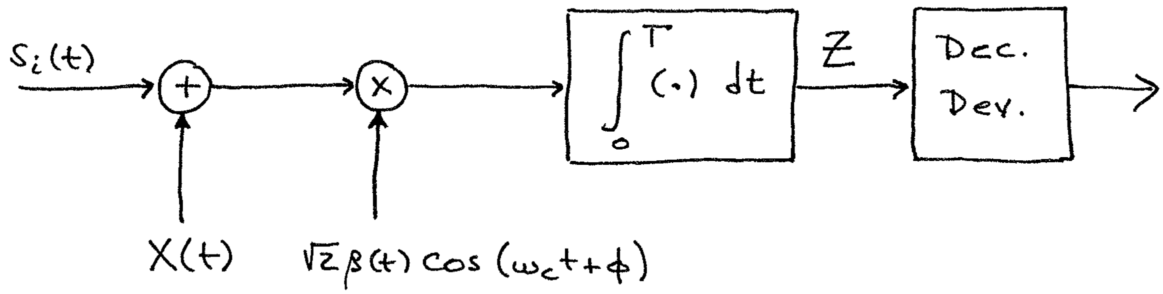
$$P_{e,0} = P_{e,M-1} = Q\left(\sqrt{2A^2\mathcal{E}_\beta/N_0}\right)$$

and that

$$P_{e,i} = 2\,Q\left(\sqrt{2A^2\mathcal{E}_\beta/N_0}\right)$$

for $1 \le i \le M-2$. Find the average probability of symbol error that results if $\pi_i = 1/M$ for each i.

MBP 6.10



M-ASK of Exercise 6-2 (p 316) is

$$s_i(t) = \sqrt{2}\, A\, u_i\, \beta(t) \cos(\omega_c t + \phi) \qquad 0 \le t \le T$$

where

$$u_i \in \{-M+1, -M+3, \dots, -1, +1, \dots M-3, M-1\} \quad M \text{ even}$$

$$\in \{-M+1, -M+3, \dots, -2, 0, +2, \dots M-3, M-1\} \quad M \text{ odd.}$$

Use ML decision rule and compute error probabilities.

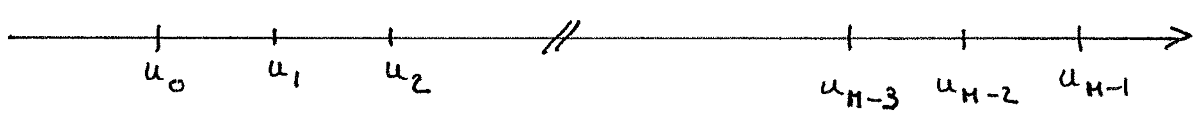
Can show that

$$Z = u_i A \|\beta\|^2 + X$$

$X \longrightarrow$ zero mean Gaussian with variance $N_0\|\beta\|^2/2$

Lets assume for either the M odd or M even cases that the symbol amplitudes are ordered s.t.

$$u_0 < u_1 < \cdots < u_{M-2} < u_{M-1}$$



The pdfs characterizing the hypotheses on z all have same variance and just differ in mean:

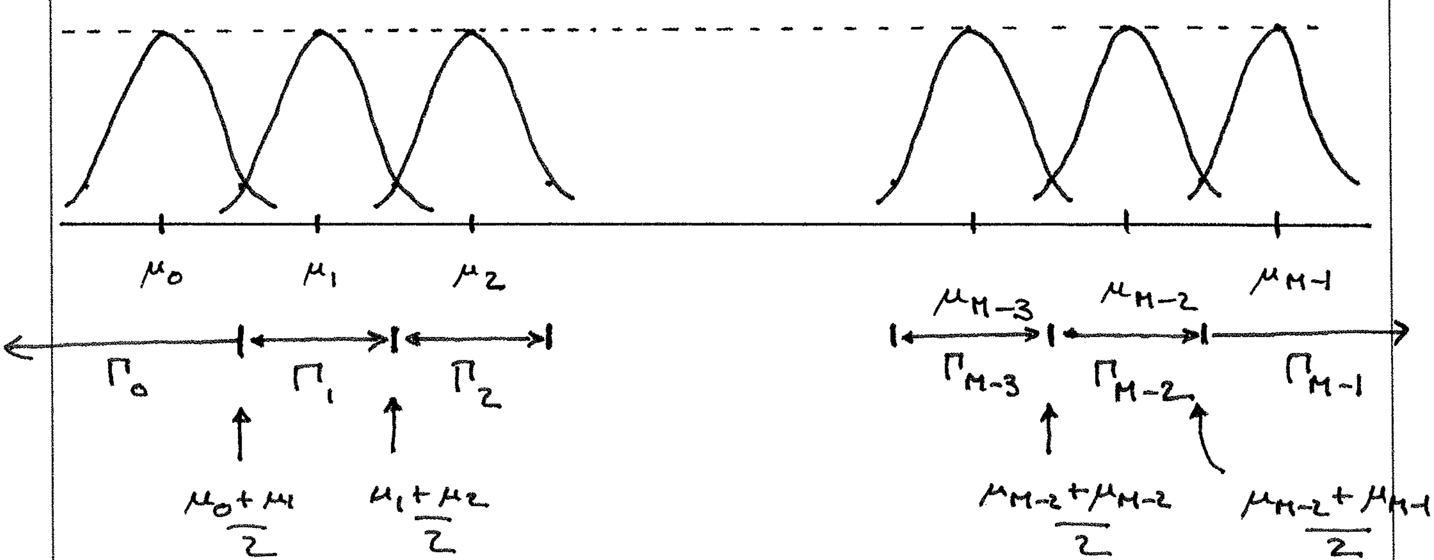
$$f_i(z) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(z-\mu_i)^2/2\sigma^2}$$

$$\mu_i = u_i A \|\beta\|^2$$
$$\sigma^2 = N_o \|\beta\|^2/2$$

The ML decision rule picks the hypothesis indexed by $\hat{k}$ given observation z where

$$\hat{k} = \underset{0 \le k \le M-1}{\arg\max} \; f_k(z)$$

The sketch below illustrates the ML rule as a partition of observation space:



Symmetry gives the boundaries shown. Also clear that

$$P_{e,i} = P_{e,1} \qquad \text{for } i = 1,2,3 \ldots M-2$$

$$P_{e,o} = P_{e,M-1} = \tfrac{1}{2} P_{e,1}$$

$$P_{e,0} = P\left(Z > \frac{\mu_0 + \mu_1}{2} \,\Big|\, H_0\right) \qquad \frac{\mu_0+\mu_1}{2} = \frac{u_0+u_1}{2} A\mathcal{E}_\beta$$

$$= \frac{-M+1-M+3}{2} A\mathcal{E}_\beta$$

$$= \frac{-2M+4}{2} A\mathcal{E}_\beta$$

$$= (-M+2) A\mathcal{E}_\beta$$

Under H_0

$$Z \sim N(\mu_0, \sigma^2)$$

$$P_{e,0} = P\left(\frac{Z-\mu_0}{\sigma} > \frac{\mu_1-\mu_0}{2\sigma} \,\Big|\, H_0\right)$$

$$= Q\left(\frac{\mu_1-\mu_0}{2\sigma}\right) \qquad \frac{\mu_1-\mu_0}{2\sigma} = \frac{A\mathcal{E}_\beta}{\sqrt{N_0\mathcal{E}_\beta/2}} = \sqrt{\frac{2A^2\mathcal{E}_\beta}{N_0}}$$

$$= Q\left(\sqrt{\frac{2A^2\mathcal{E}_\beta}{N_0}}\right) = P_{e,M-1}$$

$$\Longrightarrow \quad P_{e,i} = 2P_{e,0}$$

$$= 2Q\left(\sqrt{\frac{2A^2\mathcal{E}_\beta}{N_0}}\right) \qquad 1 \le i \le M-2$$

Avg. Prob. Error for equal priors $\pi_i = 1/M$

$$\overline{P_e} = \frac{1}{M}\sum_{i=0}^{M-1} P_{e,i}$$

$$= \frac{2Q\left(\sqrt{\frac{2A^2\mathcal{E}_\beta}{N_0}}\right)}{M} + \frac{(M-2)}{M} 2Q\left(\sqrt{\frac{2A^2\mathcal{E}_\beta}{N_0}}\right)$$

$$= \frac{2}{M}(M-1)\, Q\left(\sqrt{\frac{2A^2\mathcal{E}_\beta}{N_0}}\right)$$