Problem Set #5: Field Theory I

1. Let F be a field of characteristic $p > 0$, and fix $c \in F$. Show that the polynomial
$$f(x) = x^p - c$$
is either irreducible over F or has a root in F.

2. Suppose q is prime and consider
$$f_q(x) = x^{q-1} + x^{q-2} + \cdots + 1.$$
   (a) Suppose a prime number p divides $f_q(a)$ for some integer a. Prove that $p = q$ or $p \equiv 1 (\text{mod} q)$.
   (b) Prove there are infinitely many primes of the form $qb + 1$, where b is an integer.

3. Suppose that F is a field and I is a nontrivial ideal in $F[x]$. If $p(x)$ is irreducible and $p(x) \in I$, show that $I = (p(x))$.

4. Find the degree of the extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, where
$$\alpha = \sqrt{\frac{1+\sqrt{5}}{2}}.$$

5. Find the degree of the extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, where
$$\alpha = 2^{1/3} + 2^{5/3}.$$

6. Prove rigorously, but as concisely as possible, that if E is an extension of F and $[E : F] = p$, then $F(a) = F$ or $F(a) = E$ for all $a \in E$.

7. Suppose that $[F(\alpha) : F] = p$ and $[F(\beta) : F] = q$, where p and q are primes. Determine (with proof) all possibilities for $[F(\alpha, \beta) : F]$.