Sequences of Independent Experiments

Many random experiments can be viewed as a repetition of a subexperiment. In such cases, we can usually determine if the outcomes of any given subexperiment can affect the outcomes of the other subexperiments. When subexperiments are independent, certain analyses of the experiment are simplified.

Binomial Probability Law

Consider a random experiment with an event A. Would like to find the probability that the event A occurs exactly k times when the experiment is repeated n times. ($k \le n$).
Each performed experiment is called a Bernoulli trial, or a trial.

Let $Pr(A) = p$, $Pr(\bar{A}) = q$ $(\cancel{p \neq q})$ $(p+q=1)$, and assume the trials are independent.

Binomial Probability Law

$$\boxed{\begin{aligned} P_n(k) &= Pr(A \text{ occurs exactly } k \text{ times in } n \text{ trials}) \\ &= \binom{n}{k} p^k q^{n-k}, \quad k = 0, \ldots, n \end{aligned}}$$

where $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$, $0! = 1$

Note: Do not care about order of occurrences.
$\binom{n}{k}$ is the # ways to choose k trials out of n trials. For each choice of k trials, we want k "successes" and n-k "failures". Thus a particular sequence of k successes and n-k failures has a probability $p^k q^{n-k}$

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Ex: Two men flip three coins each. Assume all flips are fair and independent.

a) Find the probability that both men flip exactly two heads each.

Let M_{ij} = i^{th} man flips j heads
$i = 1, 2$, $j = 0, 1, 2, 3$

$$\begin{aligned}
&Pr(\text{both flip two heads}) \\
&= Pr(M_{12} \cap M_{22}) \\
&= Pr(M_{12})\, Pr(M_{22}) \\
&= P_3(2) \cdot P_3(2) \quad , \quad p = q = 1/2 \\
&= \left[\binom{3}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)\right]^2 = 0.1406
\end{aligned}$$

b) Find the probability one man flips 0 heads and one man flips 3 heads.

Pr(one flips 0 heads and one flips 3)

$$Pr((M_{10} \cap M_{23}) \cup (M_{13} \cap M_{20}))$$

$$= Pr(M_{10} \cap M_{23}) + Pr(M_{13} \cap M_{20})$$
$$= Pr(M_{10}) Pr(M_{23}) + Pr(M_{13}) Pr(M_{20})$$
$$= P_3(0) P_3(3) + P_3(3) P_3(0)$$
$$= 2 \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 \cdot \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0$$

$$= 0.0312$$

c) Find the probability one flips 0 heads and one flips 3 heads given exactly one flips 0 ~~hed~~ heads.

Conditional Probability

$$Pr((M_{10} \cap M_{23}) \cup (M_{13} \cap M_{20}) \mid (M_{10} \cap \overline{M_{20}}) \cup (\overline{M_{10}} \cap M_{20}))$$

$$= \frac{Pr((M_{10} \cap M_{23}) \cup (M_{13} \cap M_{20}))}{Pr((M_{10} \cap \overline{M_{20}}) \cup (\overline{M_{10}} \cap M_{20}))}$$

$$= \frac{2\binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0}{2\binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 \left(1-\binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3\right)}$$

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<u>Ex</u> Two riflemen fire at a target n times. Rifleman i hits the target on any given shot with probability p_i, $i=1,2$. Assume all shots are independent.

a) Find the probability that both men hit the target k times. $(0 \le k \le n)$

Pr(both men hit k times)

$= \Pr(\{1^{st}$ man hits k times$\} \cap \{2^{nd}$ man hits k times$\})$

$= \Pr(\{1^{st}$ man hits k times$\}) \Pr(\{2^{nd}$ man hits k times$\})$

$$= \binom{n}{k} p_1^k (1-p_1)^{n-k} \binom{n}{k} p_2^k (1-p_2)^{n-k}$$

b) Find the probability that both shots hit the target k times.
Consider trials with the event of interest "both shots hit the target"

$p = \Pr($ both shot hit the target $)$

$= \Pr(\{1^{st}$ man hits target$\}) \Pr(\{2^{nd}$ man hits target$\})$

$= p_1 \cdot p_2$

$$\Pr(\text{both shots hit } k \text{ times}) = \binom{n}{k} (p_1 p_2)^k (1-p_1 p_2)^{n-k}$$

c) Find the probability that at least one shot hits the target k times $(0 \le k \le n)$

Consider trials with the event "at least one hits target"

$$p = Pr(\text{at least one hits target})$$
$$= Pr(\{1^{st} \text{ man hits target}\} \cup \{2^{nd} \text{ man hits target}\})$$
$$= Pr(\{1^{st} \text{ man hits}\}) + Pr(\{2^{nd} \text{ man hits}\}) - Pr(\{1^{st} \text{man hits}\} \cap \{2^{nd} \text{ man hits}\})$$
$$= p_1 + p_2 - p_1 p_2$$

$Pr(\text{at least one shot hits target } k \text{ times})$

$$= \binom{n}{k} (p_1 + p_2 - p_1 p_2)^k \left(1 - (p_1 + p_2 - p_1 p_2)\right)^{n-k}$$

Geometric Probability Law

Consider independent trials of a random experiment where we are interested in the first occurrence of an event A. Then the probability that A occurs after m trials is given by:

Geometric Probability Law

$$\boxed{p(m) = (1-p)^{m-1} p \quad , \quad m = 1, 2, \ldots}$$

where $Pr(A) = p$

Note: The event occurs for the first time after m trials when the first $m-1$ trials are failures

Ex Flip unfair coin until first heads, where $Pr(\{H\}) = p$ and all flips are independent

~~Pr~~ Probability that first heads appears on the m^{th} trial is

$$p(m) = (1-p)^{m-1} p \quad , \quad m = 1, \ldots$$