Problem 1

(a) Find E_∞ and P_∞ for $x[n] = \left(\frac{1}{2}\right)^n \cos(\pi n)\, u[n]$.
Note that $E_\infty \triangleq \sum_{n=-\infty}^{\infty} x^2[n]$ and that

$$\cos \pi n = (-1)^n$$

$$\therefore\ x^2[n] = \left(\frac{1}{2}\right)^{2n} (-1)^{2n} = \left(\frac{1}{2}\right)^{2n}$$

$$E_\infty = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{2} = \frac{1}{1-1/4} = \frac{4}{3}.$$

Since $E_\infty < \infty$ must have $P_\infty = 0$.

(b) $x[n] = 3^n u[-n+2]$

$$X(z) = \sum_{n=-\infty}^{2} 3^n z^{-n} \qquad (m = -n)$$

$$= \sum_{m=-2}^{\infty} 3^{-m} z^{m} = 3^2 z^{-2} + 3z^{-1} + \sum_{m=0}^{\infty} (3^{-1} z)^m$$

$$= \underbrace{9z^{-2} + 3z^{-1}}_{z \neq 0 \text{ for this term.}} + \frac{1}{1 - \frac{1}{3} z} \qquad |3^{-1} z| < 1$$

$$|z| < 3. \quad \text{for this term.}$$

$\therefore$ RoC: $0 < |z| < 3$.

Can also write $X(z) = \dfrac{9}{z^2\left(1 - \frac{1}{3} z\right)} = \dfrac{9z^{-3}}{z^{-1} - 1/3}$

$$= -\frac{27 z^{-3}}{1 - 3z^{-1}}.$$

(c) Use basic LTI system property upon noting that

$$\tilde{x}(t) = x(t) - x(t - T/2)$$

$$\Rightarrow \tilde{y}(t) = y(t) - y(t - T/2)$$

$$= e^{-at} \sin(\omega_0 t) u(t) - e^{-a(t - T/2)} \sin(\omega_0 (t - T/2))\, u(t - T/2).$$

(d) Fundamental period is $\omega_0 = 2\pi/T$. Have

$$X_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn2\pi t/T}\, dt$$

Also have

$$\overline{X}_n = \frac{1}{T}\int_0^T x(t-T/2)\, e^{-j2\pi n t/T}\, dt \qquad \tau = t - T/2$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)\, e^{-j2\pi n\tau/T}\, \underbrace{e^{-j2\pi nT/2T}}_{e^{-j\pi n} = (-1)^n}\, dt$$

$$= (-1)^n X_n$$

(e) From Table $H(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$ $|z| > 1/3$ The inverse system must be st

$$H_{inv}(z)\, H(z) = 1 \quad \text{ie} \quad H_{inv}(z) = \frac{1}{H(z)} = 1 - \frac{1}{3}z^{-1}$$

The ROC of $H_{inv}(z)$ is $\mathbb{C}$ and there is only one impulse response $h_{inv}[n]$

$$h_{inv}[0] = 1 \qquad h_{inv}[n] = 0 \text{ for } n \neq 0, 1.$$
$$h_{inv}[1] = -\frac{1}{3}$$

(f) Don't have a DTFT table entry for this signal but do have a Z-transform entry.

$$F(z) = \frac{1-(e^{-10}\cos 5)\, z^{-1}}{1-(2e^{-10}\cos 5)\, z^{-1} + e^{-20} z^{-2}} \qquad |z| > e^{-10}$$

ROC includes the unit circle so

$$F(e^{j\omega}) = \frac{1 - (e^{-10}\cos 5)\, e^{-j\omega}}{1-(2e^{-10}\cos 5)\, e^{-j\omega} + e^{-20} e^{-j2\omega}}.$$

Problem 2

(a) False as stated since $F(s)\Big|_{s=j\omega}$ only gives CTFT if ROC includes the unit circle. Obviously, could add this condition and then claim would be true.

(b) False in general.

Take the signal $g(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$ which has FS coefficients

$$G_n = \frac{1}{T} \quad \forall n.$$

If we add the condition that $g(t)$ has finite energy over one period (equivalently, periodic $g(t)$ has finite power) then

$$\int_0^T |g(t)|^2 dt < \infty \implies \sum_{n=-\infty}^{\infty} |G_n|^2 < \infty$$

$$\implies |G_n| \to 0 \quad \text{as} \quad |n| \to \infty$$

then it would be true.

(c)
$$y(t) = \int_{-\infty}^{\infty} (t-\tau)\tau^2 u(t-\tau)\, x(\tau)\, d\tau$$

$$= \int_{-\infty}^{t} (t-\tau)\tau^2 x(\tau)\, d\tau.$$

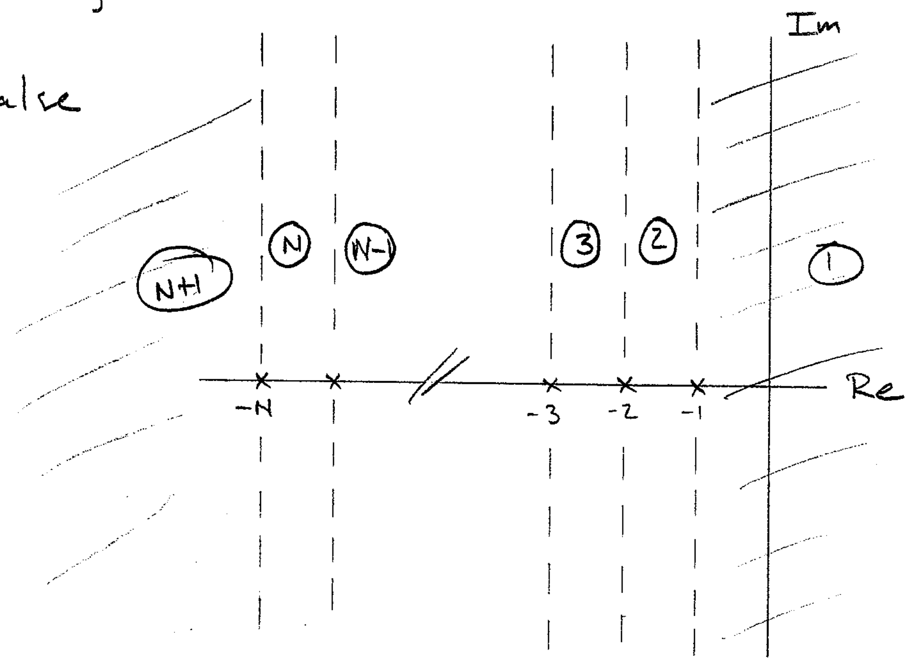
T (c-i) Clearly shows system is causal.

F (c-ii) Integrand not quite in form of a convolution sum due to τ^2 term! Therefore not time-invarient.

T (c-iii) Is linear since integrals are linear.

(d) False. Take any nonlinear and cascade with inverse function (if exists) to produce identity system, which is linear.

(e) False



There are N+1 possible ROCs. Hence N+1 possible inverse transforms.

Problem 3

$$H(z) = \frac{\alpha(1-\beta z^{-1})}{1-(1/2)z^{-1}} \qquad |z| > 1/2$$

(a) Note that $x_1[k] = (-1)^k u[k] \longleftrightarrow \frac{1}{1+z^{-1}} \quad |z|>1$
and

$$x_2[k] = u[k] \longleftrightarrow \frac{1}{1-z^{-1}} \quad |z|>1.$$

If $y_i[k]$ is output due to $x_i[k]$ $i=1,2$ then

$$Y_1(z) = \frac{\alpha(1-\beta z^{-1})}{1-\frac{1}{2}z^{-1}} \frac{1}{1+z^{-1}} = \underbrace{\frac{A}{1-\frac{1}{2}z^{-1}}}_{\text{this will corresp. to a decaying or transient term.}} + \underbrace{\frac{B}{1+z^{-1}}}_{\text{this will corresp. to steady state.}} \quad |z|>1$$

$\therefore$ Only need B

$$B = \left.\frac{\alpha(1-\beta z^{-1})}{1-\frac{1}{2}z^{-1}}\right|_{z^{-1}=-1} = \frac{\alpha(1+\beta)}{1+\frac{1}{2}} = \frac{2}{3}\alpha(1+\beta).$$

$$\Longrightarrow \quad y_{1,ss}[n] = \frac{2}{3}\alpha(1+\beta)(-1)^n u[n].$$

Similarly

$$Y_2(z) = \frac{\alpha(1-\beta z^{-1})}{1-\frac{1}{2}z^{-1}} \frac{1}{1-z^{-1}} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-z^{-1}}.$$

Only need B for same reason as before

$$B = \left.\frac{\alpha(1-\beta z^{-1})}{1-\frac{1}{2}z^{-1}}\right|_{z^{-1}=1} = \frac{\alpha(1-\beta)}{1-\frac{1}{2}} = 2\alpha(1-\beta).$$

$$\Longrightarrow \quad y_{2,ss}[n] = 2\alpha(1-\beta)u[n].$$

N.B. The values of B can also be obtained from

$$B_{1\text{st case}} = H(z)\Big|_{z=-1}, \quad B_{2\text{nd case}} = H(z)\Big|_{z=1}.$$

this follows from the eigenfunction/eigenvalue interp. of system function.

Applying constraints #1 and #2:

$$\frac{2}{3}\alpha(1+\beta) = 10 \quad \text{and} \quad 2\alpha(1-\beta) = 0.$$

Since $\alpha = 0$ is <u>not</u> a solution must have $\beta = 1$ whence

$$\frac{2}{3}\alpha \cdot 2 = 10 \implies \alpha = \frac{30}{4} = \frac{15}{2}.$$

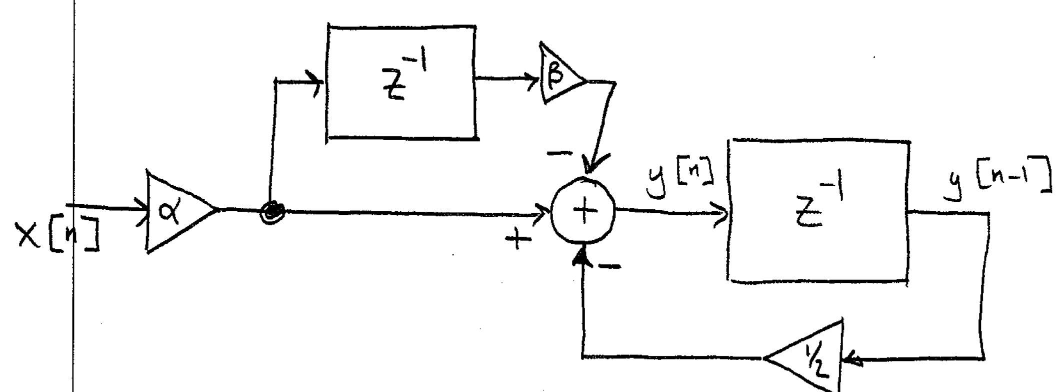
(b) Block diagram. Leave in terms of α, β in case students don't get right values.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\alpha - \alpha\beta z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

$$\implies Y(z)\left(1 - \tfrac{1}{2}z^{-1}\right) = X(z)\left(\alpha - \alpha\beta z^{-1}\right)$$

Taking inverse transform:

$$y[n] - \tfrac{1}{2}y[n-1] = \alpha x[n] - \alpha\beta x[n-1]$$



Problem 4 $Y(s) = \dfrac{2s}{s^3 + s^2 - 4s - 4}$

(a) Guess $s = 2$ is root! $2^3 + 2^3 - 4 \cdot 2 - 4 = 8 + 4 - 8 - 4 = 0.$

$$
\begin{array}{r|l}
 & s^2 + 3s + 2 \\
\hline
s-2 & s^3 + s^2 - 4s - 4 \\
 & s^3 - 2s^2 \\
\hline
 & \quad 3s^2 - 4s - 4 \\
 & \quad 3s^2 - 6s \\
\hline
 & \qquad 2s - 4 \\
 & \qquad 2s - 4
\end{array}
$$

Also easy to see
$s^2 + 3s + 2 = (s+2)(s+1)$

$\therefore$ roots of denominator are
$s = 2, -2, -1$

(b) Check

$$(s-2)(s+2)(s+1) = (s^2 - 4)(s+1)$$
$$= s^3 + s^2 - 4s - 4$$

(c) $$Y(s) = \frac{2s}{(s-2)(s+2)(s+1)}$$

$$= \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$A = \left.\frac{2s}{(s+2)(s+1)}\right|_{s=2} = \frac{4}{4 \cdot 3} = \frac{1}{3}$$

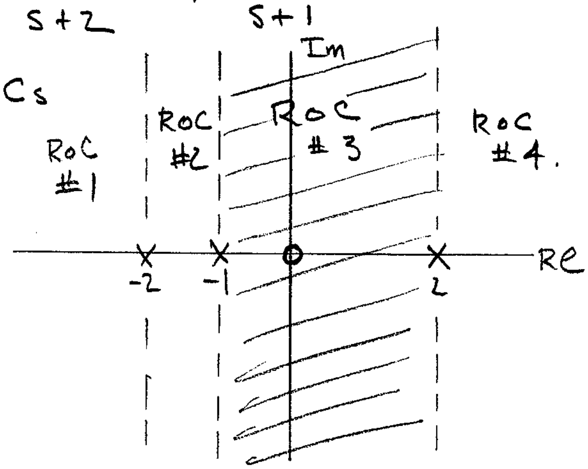
$$B = \left.\frac{2s}{(s-2)(s+1)}\right|_{s=-2} = \frac{-4}{-4 \cdot (-1)} = -1$$

$$C = \left.\frac{2s}{(s-2)(s+2)}\right|_{s=-1} = \frac{-2}{(-3)(+1)} = +\frac{2}{3}.$$

$$\Longrightarrow Y(s) = \frac{1/3}{s-2} - \frac{1}{s+2} + \frac{2/3}{s+1}$$

(d) Look at possible RoCs

$\therefore$ Four possible $y(t)$.



(e) If $y(t)$ is to be absolutely integrable the ROC must include the imag. axis ie

ROC # 3 $\quad -1 < Re(s) < 2$

To get this $y(t)$ the 1st order terms from the PFE must have ROCs as shown below:

pole @ $s=2$ $\quad Re(s) < 2 \quad \frac{1/3}{s-2} \longleftrightarrow -\frac{1}{3}e^{2t}u(-t)$.

pole @ $s=-2$ $\quad Re(s) > -2 \quad \frac{-1}{s+2} \longleftrightarrow -e^{-2t}u(t)$

pole @ $s=-1$ $\quad Re(s) > -1 \quad \frac{2/3}{s+1} \longleftrightarrow \frac{2}{3}e^{-t}u(t)$.

$$\therefore\ y(t) = -\frac{1}{3}e^{2t}u(-t) - e^{-2t}u(t) + \frac{2}{3}e^{-t}u(t).$$

(f) $\quad x(t) \longrightarrow \boxed{h(t) \leftrightarrow H(s)} \longrightarrow y(t)$

(f-i) $\quad x(t) = \delta(t) - 2e^{-2t}u(t) \longrightarrow X(s) = 1 - \frac{2}{s+2}$

$Re(s) > -2$

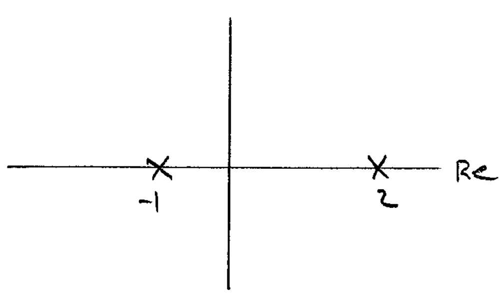
ie $\quad X(s) = \dfrac{s}{s+2} \quad Re(s) > -2.$

(f-ii) Now must have

$$H(s)X(s) = Y(s)$$

and the ROCs of H and X must overlap and their intersection must correspond to the ROC for Y(s) chosen in part (e).

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2\cancel{s}}{(s-2)\cancel{(s+2)}(s+1)}\,\frac{\cancel{s+2}}{\cancel{s}} = \frac{2}{(s-2)(s+1)}$$



pole-zero plot of H(s).

There are 3 possible RoCs for $H(s)$. To make

$$\text{RoC}_H \cap \text{RoC}_X = -1 < \text{Re}(s) < 2 = \text{RoC of } Y(s)$$

corresp. to part (e)

we must use

$$\text{RoC}_H = -1 < \text{Re}(s) < 2.$$

(g) For a fixed $x(t)$ we can get 3 different outputs $y(t)$ by picking RoC of $H(s)$ .... it corresponds to 3 different impulse responses for system. Each RoC_H will overlap with RoC_X so all signals are well defined.

But there are only 3 $y(t)$s that can be made this way and 4 possible $y(t)$s from (d).

The one left out is the anticausal $y(t)$ that comes from RoC #1 of part (d).

Problem 5

(a) Define $x_p(t) = x(t)\,p(t) = \sum_n x(nT)\,\delta(t-nT)$ and note that

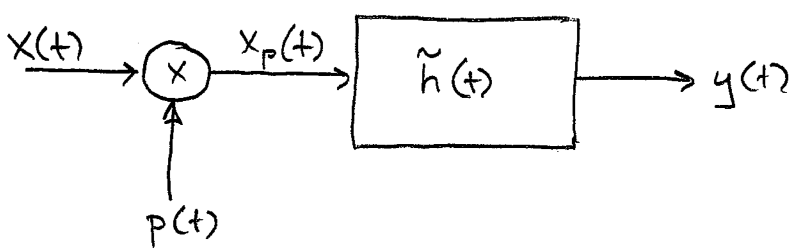
$$y(t) = \left[x_p(t) - x_p(t-T)\right] * h(t)$$

$$= \sum_n x(nT)\,h(t-nT) - \sum_m x(mT)\,h(t-T-mT)$$

$$= \sum_n x(nT)\,\underbrace{\left[h(t-nT) - h(t-T-nT)\right]}_{\tilde{h}(t-nT)}$$

where $\tilde{h}(t) = h(t) - h(t-T)$.

(b) Given $y(t) = \sum_n x(nT)\,\tilde{h}(t-nT)$ we could just recognize the equivalence to



$\Longrightarrow$ $Y(j\omega) = \tilde{H}(j\omega)\,X_p(j\omega)$ and we have a formula for

$$X_p(j\omega) = \frac{1}{T}\sum_k X(j(\omega - k\omega_s)) \qquad \omega_s = 2\pi/T$$

This formula comes from the multiplication in time / conv. in frequency property and

$$p(t) = \sum_n \delta(t-nT) \longleftrightarrow \frac{2\pi}{T}\sum_k \delta(\omega - k\omega_s) = P(j\omega).$$

$$x_p(t) = x(t)\,p(t) \longleftrightarrow X_p(j\omega) = \frac{1}{2\pi}\,X(j\omega) * P(j\omega)$$

(b-i)

$$= \frac{1}{T}\sum_k X(j(\omega - k\omega_s)).$$

Another way would be to directly transform

$$y(t) = \sum_n x(nT)\tilde{h}(t-nT)$$

$$\updownarrow$$

$$Y(j\omega) = \sum_n x(nT)e^{-jn\omega T}\tilde{H}(j\omega)$$

How do we show this is $= \frac{1}{T}\sum_k X(j(\omega - k\omega_s))$? One was would be to notice they are both periodic functions of ω of period $\omega_s = 2\pi/T$ and then prove they have the same Fourier series. This was done in a HW solution.

So have

$$Y(j\omega) = \left(\frac{1}{T}\sum_k X(j(\omega - k\omega_s))\right)\tilde{H}(j\omega).$$

Also since

$$\tilde{h}(t) = h(t) - h(t-T)$$

have

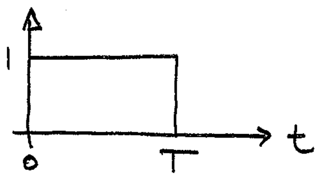
$$\tilde{H}(j\omega) = H(j\omega) - e^{-j\omega T}H(j\omega) \qquad \text{(b-ii)}$$
$$= \left(1 - e^{-j\omega T}\right)H(j\omega).$$

$$\Longrightarrow Y(j\omega) = \left(1 - e^{-j\omega T}\right)H(j\omega)\frac{1}{T}\sum_k X(j(\omega - k\omega_s))$$

(b-iii)

(c) $h(t) = u(t)$.

(c-i) Then $\tilde{h}(t) = u(t) - u(t-T)$

$=$ 

(c-ii) Taking the Fourier Transform

$$\tilde{H}(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega) - e^{-j\omega T}\left\{\frac{1}{j\omega} + \pi\delta(\omega)\right\}$$

$$= \frac{1 - e^{-j\omega T}}{j\omega} + \underbrace{\pi\delta(\omega)\left[1 - e^{-j\omega T}\right]}_{\substack{\| \\ 0}}$$

since $1 - e^{-j\omega T} = 0$ for $\omega = 0$.

$$= \frac{2e^{-j\omega T/2}}{\omega}\left[\frac{e^{+j\omega T/2} - e^{-j\omega T/2}}{j2}\right]$$

$$= T e^{-j\omega T/2} \frac{2}{\omega T} \sin(\omega T/2)$$

$$= e^{-j\omega T/2} \underbrace{T \cdot \frac{\sin(\omega T/2)}{(\omega T/2)}}$$

$= T$ @ $\omega = 0$

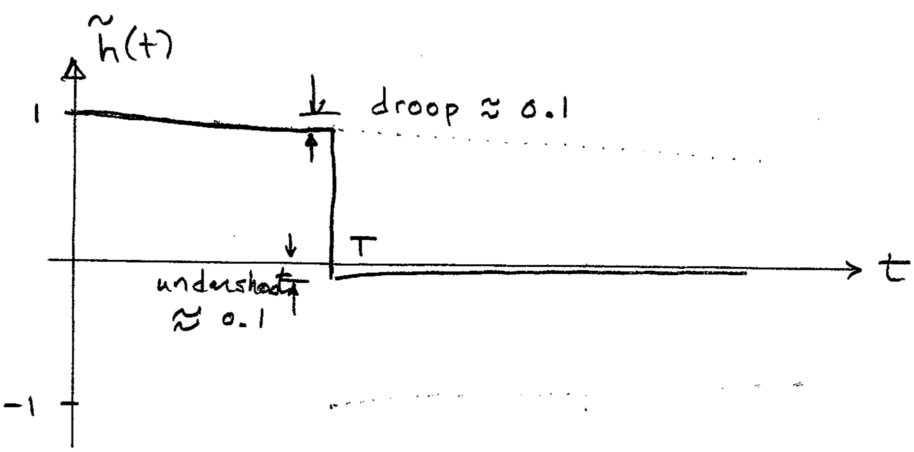
$= 0$ @ $\omega = \frac{2\pi}{T}k = \omega_s k$

$k = \pm 1, \pm 2, \ldots$

(d) $h(t) = e^{-at} u(t)$

(d-i) Let $aT = 0.1$

$$\tilde{h}(t) = e^{-at} u(t) - e^{-a(t-T)} u(t-T)$$



$$e^{-0.1} \approx 0.9$$

(d-ii)

$$\tilde{H}(j\omega) = \frac{1}{j\omega + a} - \frac{e^{-j\omega T}}{j\omega + a} = \frac{1 - e^{-j\omega T}}{j\omega + a}$$

$$= \frac{j e^{-j\omega T/2}\, 2 \sin(\omega T/2)}{j\omega + a}$$

$$|\tilde{H}(j\omega)| = \frac{2\,|\sin(\omega T/2)|}{\sqrt{a^2 + \omega^2}}$$

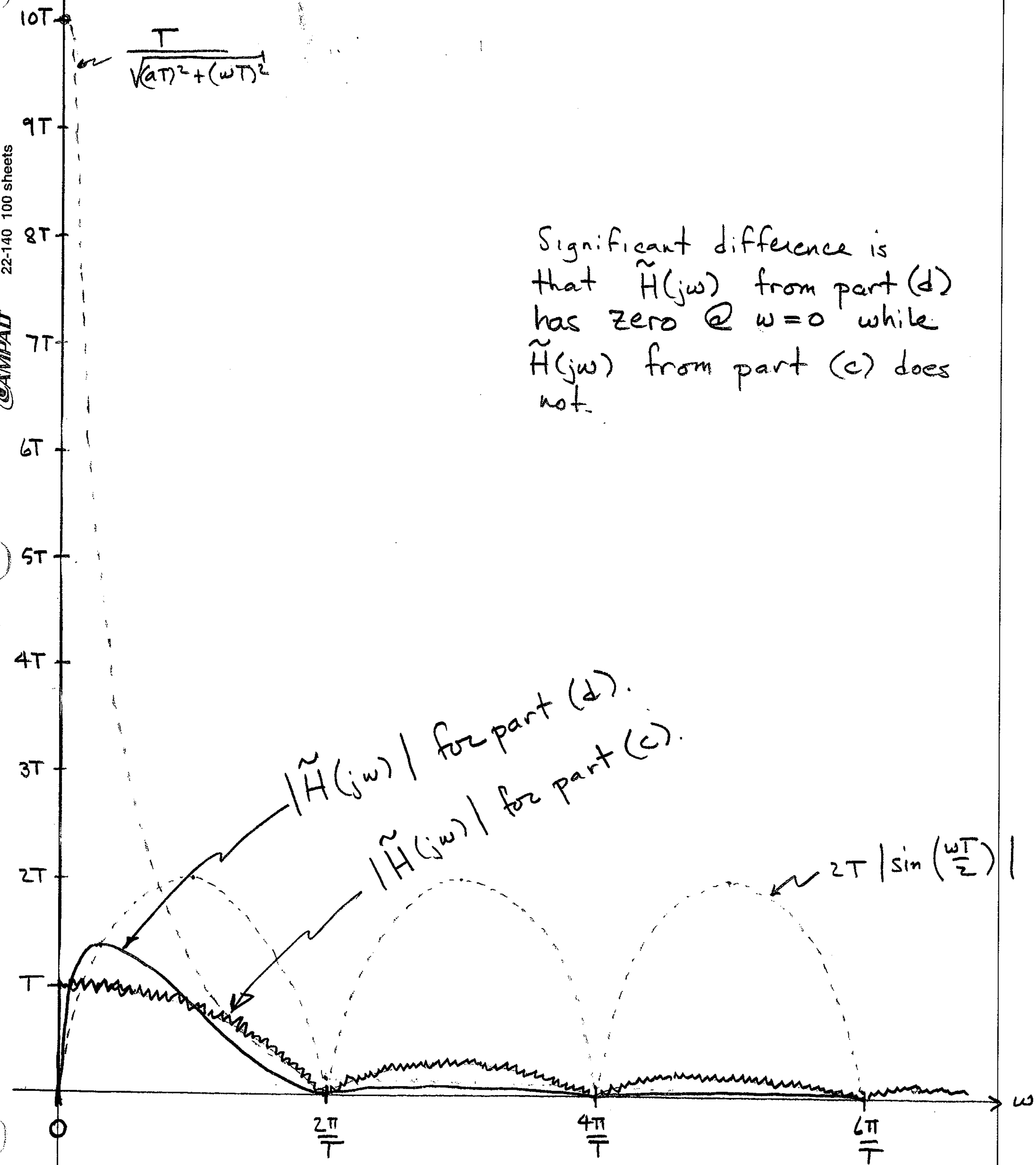
Numerator has zeros at $\omega = \frac{2\pi}{T} k = \omega_s k \quad k = 0, \pm 1, \pm 2, \ldots$

Denominator is minimum for $\omega = 0$ (but nonzero).

$$\frac{1}{\sqrt{a^2 + \omega^2}} = \frac{1}{a} \quad @ \quad \omega = 0$$

$$= 10\,T$$

Significant difference is that $\tilde{H}(j\omega)$ from part (d) has zero @ $\omega = 0$ while $\tilde{H}(j\omega)$ from part (c) does not.



(both plots symmetric in ω).

(f) Perfect reconstruction condition

(f-i) $|X(j\omega)| = 0$ for $|\omega| \geq \frac{\pi}{T}$

(f-ii) $|X(j\omega)| = 0$ for $\omega = 0$ <u>and</u> $|\omega| \geq \frac{\pi}{T}$.