(15 pts) **1.** Compute the Fourier transform of the DT signal

$$x[n] = n^2u[n-2] - n^2u[n+2]$$

(Express your answer as a linear combination of sine and/or cosine functions.)

$$x[n] = n^2u[n-2] - n^2u[n+2]$$

$$= n^2\left(u[n-2] - u[n+2]\right)$$

$$X(\omega) = -\sum_{n=-2}^{2} n^2 e^{-j\omega n}$$

$$= -4e^{j2\omega} - e^{j\omega} + e^{-j\omega} + 4e^{-2j\omega}$$

$$= \boxed{-2\cos(\omega) - 8\cos(2\omega)}$$

(15 pts) **2.** Show that the Fourier transform of the CT signal $x(t) = cos(\omega_0 t)$ is $\mathcal{X}(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$.

$$\mathcal{X}(\omega) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\pi\delta(\omega+\omega_0)+\pi\delta(\omega-\omega_0)\right]e^{-j\omega t}\,d\omega$$

$$= \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

$$= \boxed{\cos(\omega_0 t)}$$

$x[-n] = \frac{1}{g[-n]^2} = \frac{1}{(-g[n])^2} = \frac{1}{g[n]^2}$

$x[n]$ is real & even

(15 pts) **3.** Given is a DT signal $x[n] = \frac{1}{g[n]^2}$ where $g[n]$ is a pure imaginary signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{cos\omega}$. Explain why Bob's answer is wrong.

$\mathcal{X}(\omega)$ is imag & even

$x[n]$ is real & even so $\mathcal{X}(\omega)$ must be real & even

$\therefore$ Bob is wrong

b) Alice says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{1}{sin\omega}$. Could Alice be right? Explain.

$x[n]$ is real & even so $\mathcal{X}(\omega)$ must be real & even

$\mathcal{X}(\omega)$ is real & odd

$\therefore$ Alice is wrong

c) Devin says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{1}{\omega^2}$. Could Devin be right? Explain.

Devin is wrong because the F.T. of any D.T. Signal is periodic.

4. A discrete-time LTI system has frequency response

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}.$$

(15 pts) a) Derive a difference equation relating the input and the output of this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$Y(\omega) = H(\omega) X(\omega) = \frac{2X(\omega)}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

$$Y(\omega)\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right) = 2X(\omega)$$

$$Y(\omega) - \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-2j\omega}Y(\omega) = 2X(\omega)$$

$$\therefore \boxed{y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]}$$

(10 pts) b) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$?

$$X(\omega) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$Y(\omega) = H(\omega)X(\omega)$$

$$= \frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}} \cdot \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{16}{(e^{-j\omega})^2-6(e^{-j\omega})+8} \cdot \frac{4}{4-e^{-j\omega}}$$

$$= \frac{16}{(e^{-j\omega}-2)(e^{-j\omega}-4)} \cdot \frac{-4}{(e^{-j\omega}-4)}$$

$$= \boxed{\frac{-64}{(e^{-j\omega}-2)(e^{-j\omega}-4)^2}}$$

(15 pts) b) Find the unit impulse response of this system.

$$H(\omega) = \frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}} \qquad h[n] = \mathcal{F}^{-1}\left[H(\omega)\right]$$

$$= \frac{16}{(e^{-j\omega})^2-6(e^{-j\omega})+8} \qquad A(x-2)+B(x-4)=16$$

$x=2,\ -2B=16,\ B=-8$

$x=4\ \ 2A=16,\ A=8$

$$= \frac{16}{(e^{-j\omega}-2)(e^{-j\omega}-4)}$$

$$= \frac{-8}{(e^{-j\omega}-2)} + \frac{8}{(e^{-j\omega}-4)}$$

$$= \frac{-2}{(1-\frac{1}{4}e^{-j\omega})} + \frac{4}{(1-\frac{1}{2}e^{-j\omega})}$$

$$h[n] = \boxed{-2\cdot\left(\tfrac{1}{4}\right)^n u[n] + 4\left(\tfrac{1}{2}\right)^n u[n]}$$

(20 pts) **5.** Use the definition of the Fourier transform (*not* the properties listed in the table) to prove the following Fourier transform property.

$$x(at+b) \xrightarrow{F} \frac{e^{j\omega\frac{b}{a}}}{-a}\mathcal{X}\left(\frac{\omega}{a}\right) \text{ for any } a, b \text{ real numbers with } a < 0.$$

$$\frac{e^{j\omega\frac{b}{a}}}{-a} X\left(\frac{\omega}{a}\right)$$

$$\mathcal{F}\left[x(at+b)\right] = \int_{-\infty}^{\infty} x(at+b)\, e^{-j\omega t}\, dt$$

Let $\tau = at+b$

$d\tau = a\,dt$

$dt = \frac{d\tau}{a}$

$t = \frac{\tau - b}{a}$

$t = \infty,\ \tau = -\infty$

$t = -\infty,\ \tau = \infty$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\frac{\tau-b}{a}}\, d\tau$$

$$= \frac{e^{j\omega\frac{b}{a}}}{-a}\underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\frac{\tau}{a}}\, d\tau}_{X\left(\frac{\omega}{a}\right)}$$

Flip bounds of integration

$$= \boxed{\frac{e^{j\omega\frac{b}{a}}}{-a} X\left(\frac{\omega}{a}\right)}$$