31 MARCH 2012

Generating Functions

Motivation: $1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$

we recognize: (1) geometric series

(2) equals $\frac{1}{1-x}$.

meaning?

If $|x| < 1$. then,

1. $1 + x + x^2 + \dots$ has "meaning" (converges)
2. the value of this series agrees with $\frac{1}{1-x}$

If $|x| \geq 1$, both fails (series no longer converges)

$\Rightarrow$ aim: converting series into function for small value of x)

set-up: Given a sequence $\{a_n\}$

we turn the sequence into a formal power series,

(formal because expression of the series get to have any meaning)

$$G_a(x) = \sum_{n=0}^{\infty} a_n x^n$$

Q: Can we turn the series into a rational function? (as in geometric series?)

A: Yes, if ; (a) you are lucky

(b) ~~an = n~~ $\{a_n\}$ obeys a linear recurrence of finite order.

lucky cases (1) Geometric series (we knew the answer)

(2) $a_n = n$

$\hookrightarrow$ $G_a(x) = 0 + 1x + 2x^2 + 3x^3 + \dots$

$= \sum_{n=0}^{\infty} \cancel{n a_n}\; n x^n$

constant $= x \sum_{n=0}^{\infty} n x^{n-1} = x \sum_{n=0}^{\infty} (x^n)' = x \left(\sum_{n=0}^{\infty} x^n \right)'$ (- kinda look like derivative)

by pow. theorem

$= x \left(\frac{1}{1-x} \right)'$

$= x \left(\frac{1}{(1-x)^2} \right) = \frac{x}{(1-x)^2}$ ✓

Lesson: we can make $n x^n$ disappear by cleverly using the derivatives

Ex$_2$: $a_n = n^2$

$$G_a(x) = \sum_{n=0}^{\infty} n^2 x^n = \sum_{n=0}^{\infty} n(nx^n) = x\sum n(x^n)'$$

$$= x\sum (nx^n)' = x\left(\sum nx^n\right)' = x\left(x\sum nx^{n-1}\right)'$$

$$= x\left(x\left(\sum x^n\right)'\right)' = x\left(x\left(\frac{1}{1-x}\right)'\right)'$$

$$= x\left(\frac{x}{(1-x)^2}\right)' = x \cdot \frac{(1-x)^2 - x \cdot 2 \cdot (1-x)\cdot(-1)}{(1-x)^4}$$

$$= \frac{x}{(1-x)^3}(1+x)$$

Ex$_3$: $a_n = 3n+4$

$$G_a(x) = \sum_{n=0}^{\infty}(3n+4)x^n = 3\sum_{n=0}^{\infty} x^n n + 4\sum_{n=0}^{\infty} x^n$$ (use ex 0, 1 to find soln)

$$= 3\frac{x}{(1-x)^2} + 4\frac{1}{1-x}$$

Generating Functions Using Recurrence

Ex $a_n = a_{n-1} + a_{n-2}$ $\quad a_0 = 0 \quad a_1 = 1$

$$G_a(x) = \sum_{n=0}^{\infty} a_n x^n$$ → we know, so;

$$= \sum_{n=2}^{\infty} a_n x^n + 0 \cdot 1 + 1 \cdot x$$ replacing a_n (so that a_{n-1}, a_{n-2} makes sense)

$$= \sum (a_{n-1} + a_{n-2})x^n + x$$

$$= \boxed{\sum (a_{n-1})x^n} + \sum (a_{n-2})x^n + x$$

↳ how can we make this look like the "original" ($a_n x^n$)

$$= x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} + x$$

replacing $m = n-1$ and $p = n-2$ . . .

$$= x\sum_{m=1}^{\infty} a_m x^m + x^2\sum_{p=0}^{\infty} a_p x^p + x$$ we sought this

$$= x\left(G_a(x) - a_0\right) + x^2\left(G_a(x)\right) + x$$

now we can see.

$a_0 = 0$ (initial cond.)

$$G_a(x) = x\,(G_a(x) - a_0) + x^2 G_a(x) + x$$

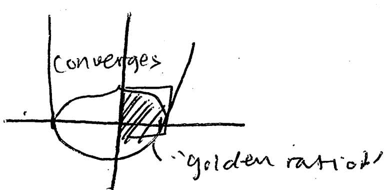
$$G_a(x)\,[1 - x - x^2] = x$$

$$G_a(x) = \frac{x}{1-x-x^2}$$

Meaning: if $|x| << 1$, then $\sum_{n=0}^{\infty} a_n x^n$ converges to the value $\frac{x}{1-x-x^2}$.

Q: How small does this x need to be?

note: $1-x-x^2$ has the root $-\frac{1}{2} \pm \sqrt{\frac{1}{4}+1} = \frac{-1 \pm \sqrt{5}}{2}$



"whichever side from zero that is shorter is the convergent radius"

Let δ = the minimum root of denominator;

then, $\sum_{n=0}^{\infty} a_n x^n = \frac{x}{1-x-x^2}$; exactly if $|x| < \delta$

<u>explicit formula for a_n:</u>

solve $a_n = a_{n-1} + a_{n-2}$ $\quad a_0 = 0 \quad a_1 = 1$

from prev. lesson, we know:

$$X(\lambda) = \lambda^2 - \lambda - 1$$

$$\lambda = \frac{1}{2} \pm \sqrt{\frac{1}{4}+1} = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow a_n = b_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + b_2 \left(\frac{1 \mp \sqrt{5}}{2}\right)^n$$

Applying initial conditions

$$b_1 + b_0 = 0$$

$$b_1\left(\frac{1+\sqrt{5}}{2}\right) + b_0\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$b_1 = \frac{1}{\sqrt{5}} \qquad b_0 = -\frac{1}{\sqrt{5}}$$

$$\Rightarrow a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Consider $G_a(x)$:

$$G_a(x) = \sum_{n=0}^{\infty} a_n x^n = \sum \left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \right) x^n$$

<u>NOTE</u>

$\downarrow$ as $n \to \infty$ "disappears"?

The dominant term of the expression is $(\frac{1+rs}{s})^n$

and we should plan to negate this term with x^n

for the series to ~~have~~ take on a meaning.

In other words, we need x such that

$$\left(\frac{1+\sqrt{5}}{2}\right)^n \cdot x^n \longrightarrow 0 \quad \text{as } n \to \infty.$$

or

$$\left(\frac{1+\sqrt{5}}{2} \cdot x\right) < 1 \qquad (\text{since } (\tfrac{1+\sqrt{5}}{2} \cdot x)^n \ldots$$

this says:

$$|x| < \frac{2}{1+\sqrt{5}}$$

in the previous quest for finding convergent radius, we saw that

$$|x| < \frac{-1+\sqrt{5}}{2} \ldots$$

comparing

$$\frac{2}{1+\sqrt{5}} \overset{?}{=} \frac{-1+\sqrt{5}}{2} \quad \Big| \begin{array}{l} \cdot\, 2 \\ \cdot\, 1+\sqrt{5} \end{array}$$

$$4 = (-1+\sqrt{5})(1+\sqrt{5}) = 4 \quad \checkmark$$

we show that x must be restricted

We arrived at finding the convergent radius by two means;

(a) we used the recursion + generating function

$\rightsquigarrow \frac{1}{1-x-x^2}$ and studied the poles

(b) we looked at the characteristic equations

studied inverses of the characteristic roots

Note: $\chi(\lambda) = \lambda^2 - \lambda - 1$

when evaluated for $\lambda = \frac{1}{\lambda}$;

$$\chi(\lambda) = \frac{1}{\lambda^2} - \frac{1}{\lambda} - 1 = \frac{1}{\lambda^2}\left(1 - \lambda - \lambda^2\right)$$ "inverse of the gen function = what we graphed...?)

FACT: For a homogeneous recurrence, the characteristic equation $\chi(\lambda)$, when evaluated at $\frac{1}{x}$ gives the denominator of $G_a(x)$

Ex. Q. Want to pay \$r using \$1, 2, 5 dollars

A: if order of payment does not matter

denote payment of \$1, \$2, or \$5 by monomial

x^1, x^2, x^5

So x^a = pay \$1 a times

$(1+x+x^2+\dots)(1+x^2+x^4+x^6+\dots)(1+x^5+x^{10}+\dots)$

has r-th term $\square \cdot x^r$ ← # of ways to pay \$r

for ex: r = 6

$x^6 \cdot 1 \cdot 1 + x^4 \cdot x^2 \cdot 1 + x^2 \cdot (x^2)^2 \cdot 1 + x \cdot 1 \cdot x^5$
$+ 1 \cdot x^6 \cdot 1 \dots$

We are looking a number ⇒ r^{th} coefficient in the power series to $\frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^5}$ a_n

$\frac{1}{1-x} \Leftrightarrow 1+x+x^2+\dots$, $\frac{1}{1-x^2} \Leftrightarrow 1+x^2+x^4+\dots$, $\frac{1}{1-x^5} \Leftrightarrow 1+x^5+x^{10}+\dots$

If order did matter

r = 6

111111 ; 11112 , ... , 21111 ; (5 cases)

1122 , ... , 2211 $\binom{4}{2}$ cases

222 , 15 , ... , 51 (2 cases)

challenge: explain why we are looking for the r^{th} coefficient in the power series to

$$\frac{1}{1-x-x^2-x^5}$$ (instead of mult, we subtract).