1. Let \( f_n \in L^1(0,1) \) such that \( f_n \to f \) in \( L^1 \). Show there exists \( M \in \mathbb{N} \) such that
\[
\limsup_{n \to \infty} \int_{\{|f_n| > M\}} |f_n| = 0.
\]

2. Let \((X, \mathcal{M}, \mu)\) be a measure space, \( f_n \) and \( f \) are \( \mu \)-measureable functions on \( X \).
   
   (a) Prove \( f_n \to f \) in measure implies there is a subsequence \( \{f_{n_k}\}_k \) converging to \( f \) a.e.
   
   (b) Suppose now that \( f_n \to f \) a.e. Prove or disprove that \( f_n \) converges to \( f \) in measure, under the following hypotheses:
   
   i. \( \mu(X) < \infty \).
   
   ii. \( \mu \) is sigma-finite.
   
   iii. No additional assumptions on our space.

3. Let \((X, \mathcal{M}, \mu)\) be a sigma-finite measure space and let \( f \in L^p(\mu), 1 \leq p < \infty \). Show
\[
\int_X |f|^p d\mu = p \int_{-\infty}^{\infty} \lambda^{p-1} \mu(\{|f| > \lambda\}) d\lambda.
\]

Where did you use sigma-finiteness?

4. Two real-valued functions \( f \) and \( g \) defined on \([0,1]\) are said to be comonotone if:
\[
(f(x) - f(y))(g(x) - g(y)) \geq 0
\]
for all \( x, y \in [0,1] \). Suppose the two functions are Lebesgue measurable. Prove that:
\[
\left( \int_0^1 f(t) dt \right) \left( \int_0^1 g(t) dt \right) \leq \int_0^1 f(t) g(t) dt.
\]

5. Let \( f \in L^2(\mathbb{R}) \) and let \( f_0(x) = xf(x) \). Show that \( \|f\|_1 \leq (8\|f\|_2 \|f_0\|_2)^{1/2} \).
   (Hint: consider \( \{|x| > a\} \) and \( \{|x| \leq a\} \).)
6. Let $g \in L^p(\mathbb{R}), 1 \leq p < \infty$ and $f(x) = e^{-|x|}$. Show where you use Fubini’s and Tonelli’s Theorems respectively to prove
\[
\int_{-\infty}^{\infty} (f * g)^p(x)dx = \frac{2}{p} \int_{-\infty}^{\infty} g(x)^p dx.
\]

7. Let $\mathcal{A}$ be the $\sigma$-algebra in $\mathbb{R}^2$ generated by the family of sets \{(x, y) : x \geq r_j, y \in [r_{2j}, r_{2j+1})\}, where $r_j \in \mathbb{Q}$ for $j = 1, 2, 3$. Prove or disprove $\mathcal{A}$ is the Borel $\sigma$-algebra on $\mathbb{R}^2$.

8. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $f_n \to f$ in $L^p, 1 \leq p < \infty$. Fix $\epsilon > 0$ and show there exists $\delta = \delta(\epsilon) > 0$ such that $\forall A \in \mathcal{M}$ with $\mu A < \delta$, we have $\int_A |f_n| < \epsilon$.

9. (Test 2-2) Let $f \in L^1([0, 1])$. Prove that the function
\[
G(t) = \int_0^1 \cos(tf(x))dx
\]
is differentiable with respect to $t$. 

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