5. (a) Show $\hat{f}(\xi)$ is continous.

take $\xi_n \to \xi$ as $n \to \infty$

$|e^{-i\xi_n x} f(x)| \leq |f| \in L^1$

Apply DCT.

$$\lim_{n\to\infty} \hat{f}(\xi_n) = \int \lim_{n\to\infty} e^{-i\xi_n x} f(x)\, dx = \int e^{-i\xi x} f(x)\, dx = \hat{f}(\xi).$$

(b) Show $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$

1° $f \in C_0(\mathbb{R}^n)$

$$\hat{f}(\xi) = \int_{\mathbb{R}} \frac{e^{-i\xi x} - e^{-i\xi x} \cdot e^{-i\pi}}{2} f(x)\, dx.$$

$$= \frac{1}{2}\left[\int_{\mathbb{R}} e^{-i\xi x} f(x)\, dx - \int_{\mathbb{R}} e^{-i(\xi x + \pi)} f(x)\, dx\right]$$

$$= \frac{1}{2}\left[\int_{\mathbb{R}} e^{-i\xi x} f(x)\, dx - \int_{\mathbb{R}} e^{-i\xi y} f\left(y - \frac{\pi}{\xi}\right) dy\right]$$

$$= \frac{1}{2} \int_{\mathbb{R}} e^{-i\xi x}\left(f(x) - f\left(x - \frac{\pi}{\xi}\right)\right) dy.$$

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = \frac{1}{2} \int_{\mathbb{R}} \lim_{|\xi|\to\infty} e^{-i\xi x}\left(f(x) - f\left(x - \frac{\pi}{\xi}\right)\right) dy = 0.$$

2° $f \in L^1(\mathbb{R})$.

$$|\hat{f}(\xi)| \leq |\hat{f}(\xi) - \hat{g}(\xi)| + |\hat{g}(\xi)|, \quad g \in C_0(\mathbb{R}).$$

$$\leq \|f - g\|_1 + |\hat{g}(\xi)|$$

$$\leq \varepsilon + |\hat{g}(\xi)|. \quad (\text{we can choose some } g \text{ s.t } \|f-g\|_1 < \varepsilon)$$

$\lim_{|\xi|\to\infty} |\hat{f}(\xi)| \leq \varepsilon$ $\quad$ ε is arbitrary.

$$\Rightarrow \lim_{|\xi|\to\infty} |\hat{f}(\xi)| = \lim_{|\xi|\to\infty} \hat{f}(\xi) = 0.$$

6. $f * f = f$.

by #2 $\Rightarrow$ $\hat{f} \cdot \hat{f} = \hat{f}$

$\Rightarrow \hat{f}(\hat{f} - 1) = 0$

$\hat{f} = 0$ or $\hat{f} = 1$

by #5, $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$ $\Rightarrow \hat{f} = 0$

by inverse Thm:

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi)\, d\xi = 0.$$

9. $f \in L^1(\mathbb{R})$.

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(4x)\, f(x+y)\, dx\, dy.$$

$$= \int_{\mathbb{R}} f(4x)\, dx \int_{\mathbb{R}} f(x+y)\, dy.$$

$$= \frac{1}{4}\int_{\mathbb{R}} f(z)\, dz \int_{\mathbb{R}} f(x+y)\, dy = \frac{1}{4}\left(\int_{\mathbb{R}} f(x)\, dx\right)^2 = 1$$

$$\Rightarrow \int_{\mathbb{R}} f(x)\, dx = 2 \text{ or } -2.$$

13.

(a) $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi\rangle} f(x)\, dx$

$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^n} |e^{-i\langle x,\xi\rangle}| |f(x)| dx = \|f\|_1$

when $\xi = 0$. $\hat{f}(0) = \int_{\mathbb{R}^n} f\, dx = \int_{\mathbb{R}^n} |f|\, dx = \|f\|_1$

by definition of $\|\cdot\|_\infty$

$\|\hat{f}\|_\infty = \|f\|_{L^1} = \hat{f}(0)$.

(b) f is continuous at 0,

<u>Claim</u>: $f(x) = \hat{\hat{f}}(x)$ if $f(x)$ is continuous at x

Assume the claim is true.

$$f(0) = \int_{\mathbb{R}^n} \underbrace{e^{-2\pi i\langle 0,\xi\rangle}}_{=1} \int_{\mathbb{R}^n} e^{-2\pi i\langle x,\xi\rangle} f(x)\, dx\, d\xi.$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-2\pi i \langle x,\xi\rangle} f(x)\, dx \right) d\xi.$$

$$= \int_{\mathbb{R}^n} \hat{f}(\xi)\, d\xi = \int_{\mathbb{R}^n} |\hat{f}(\xi)|\, d\xi = \|\hat{f}\|_{L^1}$$

Then we are done.

I don't know how to prove the claim.

\# 12 $f \in L^1$, $\hat{f} \in L^1$

$\varphi(x) = (4\pi)^{-n/2} e^{-|x|^2/4}$, $\varphi_\varepsilon(x) = \varepsilon^{-n} \cdot (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4\varepsilon^2}}$,

$$\int_{\mathbb{R}^n} \hat{\varphi}_\varepsilon(\xi) \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi = \int_{\mathbb{R}^n} e^{2\pi i \langle x,\xi\rangle} \hat{\varphi}_\varepsilon(\xi) \left[\int_{\mathbb{R}^n} e^{-2\pi i \langle y, \xi \rangle} f(y) dy \right] d\xi$$

by Fubini's Thm.

$$= \int_{\mathbb{R}^n} f(y) \left[\int_{\mathbb{R}^n} e^{2\pi i \langle x-y, \xi \rangle} \hat{\varphi}_\varepsilon(\xi) d\xi \right] dy.$$

$$\overset{*}{=} \int_{\mathbb{R}^n} f(y) \varphi_\varepsilon(x-y) dy = f * \varphi_\varepsilon(x).$$

we know $f * \varphi_\varepsilon \to f$ in $L^1(\mathbb{R}^n)$ as $\varepsilon \to 0$

$\exists$ $f * \varphi_{\varepsilon_n} \to f$ a.e as $\varepsilon_n \to 0$ $(n \to \infty)$

$$\lim_{n \to \infty} \int \hat{\varphi}_{\varepsilon_n}(\xi) \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi = \lim_{n\to\infty} f * \varphi_{\varepsilon_n} = f \text{ a.e}$$

$|\hat{\varphi}_{\varepsilon_n}(\xi) \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle}| \le c \cdot |\hat{f}(\xi)| \in L^1$, c is a constant

by DCT

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} \hat{\varphi}_{\varepsilon_n}(\xi) \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} = \int_{\mathbb{R}^n} \lim_{n \to \infty} \hat{\varphi}_{\varepsilon_n}(\xi) \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle}$$

$$= \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi = f(x) \text{ a.e}$$

$$\widehat{e^{-\lambda^2 |\cdot|^2}}(\xi) = \frac{\pi^{\frac{n}{2}}}{\lambda^n} e^{-\frac{\pi^2}{\lambda^2}|\xi|^2}.$$

* easy to verify:

$$\int_{\mathbb{R}^n} e^{2\pi i \langle x, \xi \rangle} \hat{\varphi}_\varepsilon(\xi) d\xi = \varphi_\varepsilon(x).$$

4.

(a).

$$(Tf)(x_n) - (Tf)(x) = \int_0^1 [G(x_n, y) - G(x, y)] f(y) dy$$

$G(x, y)$ is cont on $[0,1] \times [0,1]$

then $\exists M_1 < \infty$ s.t $|G(\cdot, y)| \leq M_1$

hence $G(\cdot, y) - G(\cdot, y) \in L^2([0,1])$

$|(Tf)(x_n) - (Tf)(x)| \leq \| G(x_n, y) - G(x, y) \|_{L^2} \|f\|_{L^2}$

hence $\lim_{n\to\infty} |(Tf)(x_n) - (Tf)(x)| = 0 \Rightarrow Tf \in C([0,1])$

$|Tf| \leq \int_0^1 |G(x,y)| |f(y)| dy$.

$G(x, y)$ is cont. on $[0,1] \times [0,1]$

$\exists M < \infty$ s.t $|G(x,y)| < M$, $M \in L^2([0,1])$

$|Tf| \leq \int_0^1 M |f(y)| dy \leq \|M\|_{L^2} \|f\|_{L^2} = M \|f\|_{L^2}$.

(b)