this chapter we have derived and examined many of these properties. Among them are two that have particular significance for our study of signals and systems. The first is the convolution property, which is a direct consequence of the eigenfunction property of complex exponential signals and which leads to the description of an LTI system in terms of its frequency response. This description plays a fundamental role in the frequency-domain approach to the analysis of LTI systems, which we will continue to explore in subsequent chapters. The second property of the Fourier transform that has extremely important implications is the multiplication property, which provides the basis for the frequency-domain analysis of sampling and modulation systems. We examine these systems further in Chapters 7 and 8.

We have also seen that the tools of Fourier analysis are particularly well suited to the examination of LTI systems characterized by linear constant-coefficient differential equations. Specifically, we have found that the frequency response for such a system can be determined by inspection and that the technique of partial-fraction expansion can then be used to facilitate the calculation of the impulse response of the system. In subsequent chapters, we will find that the convenient algebraic structure of the frequency responses of these systems allows us to gain considerable insight into their characteristics in both the time and frequency domains.

Chapter 4 Problems

The first section of problems belongs to the basic category and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

BASIC PROBLEMS WITH ANSWERS

4.1. Use the Fourier transform analysis equation (4.9) to calculate the Fourier transforms of:

(a) $e^{-2(t-1)}u(t-1)$ **(b)** $e^{-2|t-1|}$

Sketch and label the magnitude of each Fourier transform.

4.2. Use the Fourier transform analysis equation (4.9) to calculate the Fourier transforms of:

(a) $\delta(t+1)+\delta(t-1)$ **(b)** $\frac{d}{dt}\{u(-2-t)+u(t-2)\}$

Sketch and label the magnitude of each Fourier transform.

4.3. Determine the Fourier transform of each of the following periodic signals:

(a) $\sin(2\pi t+\frac{\pi}{4})$ **(b)** $1+\cos(6\pi t+\frac{\pi}{8})$

4.4. Use the Fourier transform synthesis equation (4.8) to determine the inverse Fourier transforms of:

(a) $X_1(j\omega) = 2\pi\,\delta(\omega)+\pi\,\delta(\omega-4\pi)+\pi\,\delta(\omega+4\pi)$

(b) $X_2(j\omega) = \begin{cases} 2, & 0 \le \omega \le 2 \\ -2, & -2 \le \omega < 0 \\ 0, & |\omega| > 2 \end{cases}$

4.5. Use the Fourier transform synthesis equation (4.8) to determine the inverse Fourier transform of $X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)}$, where

$$|X(j\omega)| = 2\{u(\omega + 3) - u(\omega - 3)\},$$
$$\sphericalangle X(j\omega) = -\frac{3}{2}\omega + \pi.$$

Use your answer to determine the values of t for which $x(t) = 0$.

4.6. Given that $x(t)$ has the Fourier transform $X(j\omega)$, express the Fourier transforms of the signals listed below in terms of $X(j\omega)$. You may find useful the Fourier transform properties listed in Table 4.1.

(a) $x_1(t) = x(1 - t) + x(-1 - t)$
(b) $x_2(t) = x(3t - 6)$
(c) $x_3(t) = \frac{d^2}{dt^2}x(t - 1)$

4.7. For each of the following Fourier transforms, use Fourier transform properties (Table 4.1) to determine whether the corresponding time-domain signal is (i) real, imaginary, or neither and (ii) even, odd, or neither. Do this without evaluating the inverse of any of the given transforms.

(a) $X_1(j\omega) = u(\omega) - u(\omega - 2)$
(b) $X_2(j\omega) = \cos(2\omega)\sin(\frac{\omega}{2})$
(c) $X_3(j\omega) = A(\omega)e^{jB(\omega)}$, where $A(\omega) = (\sin 2\omega)/\omega$ and $B(\omega) = 2\omega + \frac{\pi}{2}$
(d) $X(j\omega) = \sum_{k=-\infty}^{\infty}(\frac{1}{2})^{|k|}\,\delta(\omega - \frac{k\pi}{4})$

4.8. Consider the signal

$$x(t) = \begin{cases} 0, & t < -\frac{1}{2} \\ t + \frac{1}{2}, & -\frac{1}{2} \le t \le \frac{1}{2}. \\ 1, & t > \frac{1}{2} \end{cases}$$

(a) Use the differentiation and integration properties in Table 4.1 and the Fourier transform pair for the rectangular pulse in Table 4.2 to find a closed-form expression for $X(j\omega)$.
(b) What is the Fourier transform of $g(t) = x(t) - \frac{1}{2}$?

4.9. Consider the signal

$$x(t) = \begin{cases} 0, & |t| > 1 \\ (t + 1)/2, & -1 \le t \le 1 \end{cases}.$$

(a) With the help of Tables 4.1 and 4.2, determine the closed-form expression for $X(j\omega)$.
(b) Take the real part of your answer to part (a), and verify that it is the Fourier transform of the even part of $x(t)$.
(c) What is the Fourier transform of the odd part of $x(t)$?

4.10. **(a)** Use Tables 4.1 and 4.2 to help determine the Fourier transform of the following signal:

$$x(t) = t\left(\frac{\sin t}{\pi t}\right)^2$$

(b) Use Parseval's relation and the result of the previous part to determine the numerical value of

$$A = \int_{-\infty}^{+\infty} t^2\left(\frac{\sin t}{\pi t}\right)^4 dt$$

4.11. Given the relationships

$$y(t) = x(t) * h(t)$$

and

$$g(t) = x(3t) * h(3t),$$

and given that $x(t)$ has Fourier transform $X(j\omega)$ and $h(t)$ has Fourier transform $H(j\omega)$, use Fourier transform properties to show that $g(t)$ has the form

$$g(t) = Ay(Bt).$$

Determine the values of A and B.

4.12. Consider the Fourier transform pair

$$e^{-|t|} \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{2}{1+\omega^2}.$$

(a) Use the appropriate Fourier transform properties to find the Fourier transform of $te^{-|t|}$.

(b) Use the result from part (a), along with the duality property, to determine the Fourier transform of

$$\frac{4t}{(1+t^2)^2}.$$

Hint: See Example 4.13.

4.13. Let $x(t)$ be a signal whose Fourier transform is

$$X(j\omega) = \delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5),$$

and let

$$h(t) = u(t) - u(t-2).$$

(a) Is $x(t)$ periodic?

(b) Is $x(t) * h(t)$ periodic?

(c) Can the convolution of two aperiodic signals be periodic?

4.14. Consider a signal $x(t)$ with Fourier transform $X(j\omega)$. Suppose we are given the following facts:

1. $x(t)$ is real and nonnegative.
2. $\mathfrak{F}^{-1}\{(1 + j\omega)X(j\omega)\} = Ae^{-2t}u(t)$, where A is independent of t.
3. $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi$.

Determine a closed-form expression for $x(t)$.

4.15. Let $x(t)$ be a signal with Fourier transform $X(j\omega)$. Suppose we are given the following facts:

1. $x(t)$ is real.
2. $x(t) = 0$ for $t \leq 0$.
3. $\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{R}e\{X(j\omega)\}e^{j\omega t} d\omega = |t|e^{-|t|}$.

Determine a closed-form expression for $x(t)$.

4.16. Consider the signal

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(k\frac{\pi}{4})}{(k\frac{\pi}{4})} \delta(t - k\frac{\pi}{4}).$$

(a) Determine $g(t)$ such that

$$x(t) = \left(\frac{\sin t}{\pi t}\right) g(t).$$

(b) Use the multiplication property of the Fourier transform to argue that $X(j\omega)$ is periodic. Specify $X(j\omega)$ over one period.

4.17. Determine whether each of the following statements is true or false. Justify your answers.

(a) An odd and imaginary signal always has an odd and imaginary Fourier transform.

(b) The convolution of an odd Fourier transform with an even Fourier transform is always odd.

4.18. Find the impulse response of a system with the frequency response

$$H(j\omega) = \frac{(\sin^2(3\omega)) \cos \omega}{\omega^2}.$$

4.19. Consider a causal LTI system with frequency response

$$H(j\omega) = \frac{1}{j\omega + 3}.$$

For a particular input $x(t)$ this system is observed to produce the output

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t).$$

Determine $x(t)$.

4.20. Find the impulse response of the causal LTI system represented by the *RLC* circuit considered in Problem 3.20. Do this by taking the inverse Fourier transform of the circuit's frequency response. You may use Tables 4.1 and 4.2 to help evaluate the inverse Fourier transform.

BASIC PROBLEMS

4.21. Compute the Fourier transform of each of the following signals:

(a) $[e^{-\alpha t}\cos\omega_0 t]u(t)$, $\alpha > 0$

(b) $e^{-3|t|}\sin 2t$

(c) $x(t) = \begin{cases} 1+\cos\pi t, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$

(d) $\sum_{k=0}^{\infty} \alpha^k \delta(t-kT)$, $|\alpha| < 1$

(e) $[te^{-2t}\sin 4t]u(t)$

(f) $\left[\frac{\sin \pi t}{\pi t}\right]\left[\frac{\sin 2\pi(t-1)}{\pi(t-1)}\right]$

(g) $x(t)$ as shown in Figure P4.21(a)

(h) $x(t)$ as shown in Figure P4.21(b)

(i) $x(t) = \begin{cases} 1-t^2, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

(j) $\sum_{n=-\infty}^{+\infty} e^{-|t-2n|}$

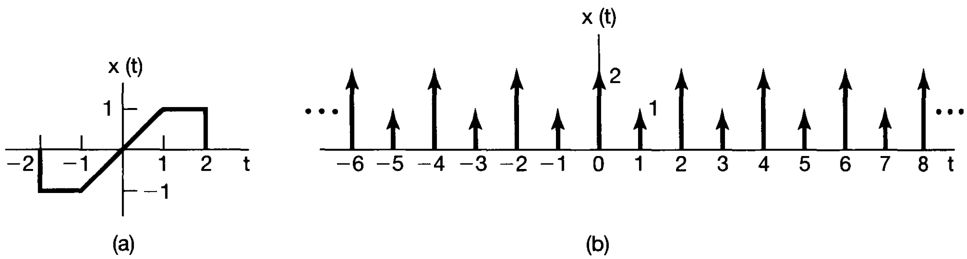

Figure P4.21

4.22. Determine the continuous-time signal corresponding to each of the following transforms.

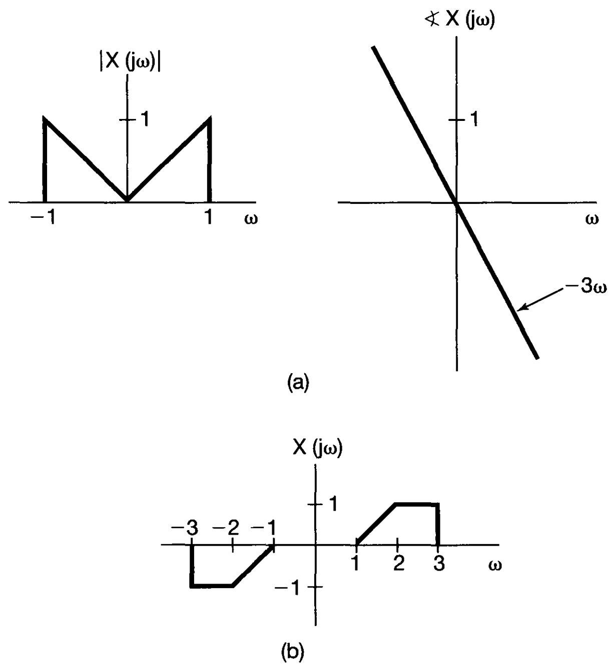

Figure P4.22

(a) $X(j\omega) = \frac{2\sin[3(\omega-2\pi)]}{(\omega-2\pi)}$
(b) $X(j\omega) = \cos(4\omega + \pi/3)$
(c) $X(j\omega)$ as given by the magnitude and phase plots of Figure P4.22(a)
(d) $X(j\omega) = 2[\delta(\omega - 1) - \delta(\omega + 1)] + 3[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)]$
(e) $X(j\omega)$ as in Figure P4.22(b)

4.23. Consider the signal

$$x_0(t) = \begin{cases} e^{-t}, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the Fourier transform of each of the signals shown in Figure P4.23. You should be able to do this by explicitly evaluating *only* the transform of $x_0(t)$ and then using properties of the Fourier transform.

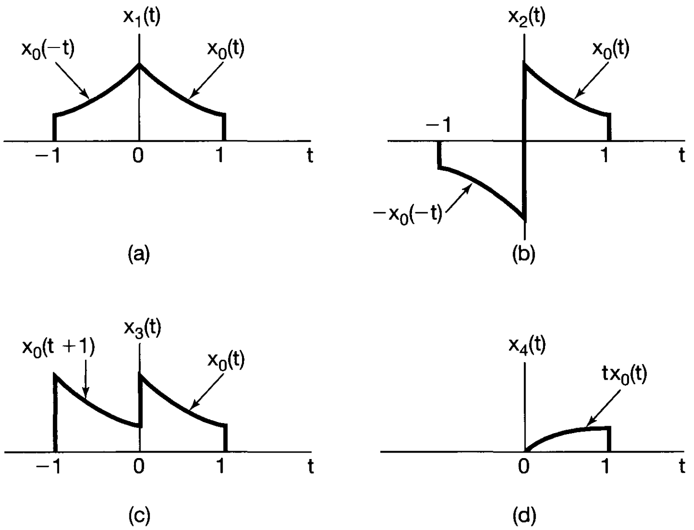

Figure P4.23

4.24. **(a)** Determine which, if any, of the real signals depicted in Figure P4.24 have Fourier transforms that satisfy each of the following conditions:
(1) $\mathcal{R}e\{X(j\omega)\} = 0$
(2) $\mathcal{I}m\{X(j\omega)\} = 0$
(3) There exists a real α such that $e^{j\alpha\omega}X(j\omega)$ is real
(4) $\int_{-\infty}^{\infty} X(j\omega)\,d\omega = 0$
(5) $\int_{-\infty}^{\infty} \omega X(j\omega)\,d\omega = 0$
(6) $X(j\omega)$ is periodic

(b) Construct a signal that has properties (1), (4), and (5) and does *not* have the others.

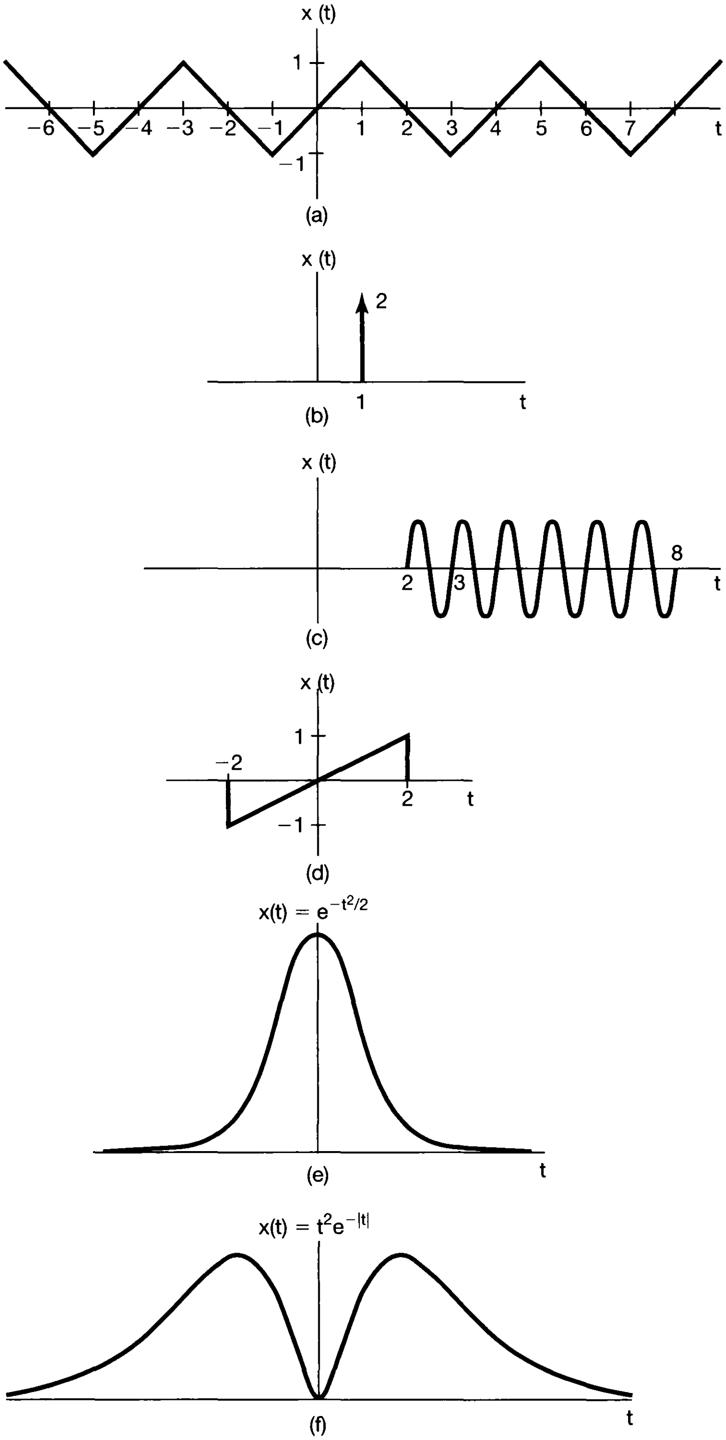
x (t)
1
−1
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7 t
(a)
x (t)
2
1 t
(b)
x (t)
2 3 8 t
(c)
x (t)
1
−2
2 t
−1
(d)
x(t) = e^{−t²/2}
t
(e)
x(t) = t²e^{−|t|}
t
(f)

Figure P4.24

4.25. Let $X(j\omega)$ denote the Fourier transform of the signal $x(t)$ depicted in Figure P4.25.

(a) Find $\sphericalangle X(j\omega)$.

(b) Find $X(j0)$.

(c) Find $\int_{-\infty}^{\infty} X(j\omega)\, d\omega$.

(d) Evaluate $\int_{-\infty}^{\infty} X(j\omega) \frac{2 \sin \omega}{\omega} e^{j2\omega}\, d\omega$.

(e) Evaluate $\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$.

(f) Sketch the inverse Fourier transform of $\mathcal{Re}\{X(j\omega)\}$.

Note: You should perform all these calculations without explicitly evaluating $X(j\omega)$.

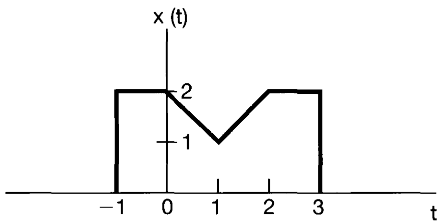

Figure P4.25

4.26. **(a)** Compute the convolution of each of the following pairs of signals $x(t)$ and $h(t)$ by calculating $X(j\omega)$ and $H(j\omega)$, using the convolution property, and inverse transforming.

(i) $x(t) = te^{-2t}u(t)$, $h(t) = e^{-4t}u(t)$

(ii) $x(t) = te^{-2t}u(t)$, $h(t) = te^{-4t}u(t)$

(iii) $x(t) = e^{-t}u(t)$, $h(t) = e^{t}u(-t)$

(b) Suppose that $x(t) = e^{-(t-2)}u(t-2)$ and $h(t)$ is as depicted in Figure P4.26. Verify the convolution property for this pair of signals by showing that the Fourier transform of $y(t) = x(t) * h(t)$ equals $H(j\omega)X(j\omega)$.

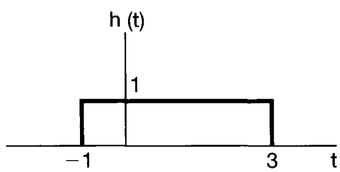

Figure P4.26

4.27. Consider the signals

$$x(t) = u(t-1) - 2u(t-2) + u(t-3)$$

and

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t-kT),$$

where $T > 0$. Let a_k denote the Fourier series coefficients of $\tilde{x}(t)$, and let $X(j\omega)$ denote the Fourier transform of $x(t)$.

(a) Determine a closed-form expression for $X(j\omega)$.

(b) Determine an expression for the Fourier coefficients a_k and verify that $a_k = \frac{1}{T}X\left(j\frac{2\pi k}{T}\right)$.

4.28. **(a)** Let $x(t)$ have the Fourier transform $X(j\omega)$, and let $p(t)$ be periodic with fundamental frequency ω_0 and Fourier series representation

$$p(t) = \sum_{n=-\infty}^{+\infty} a_n e^{jn\omega_0 t}.$$

Determine an expression for the Fourier transform of

$$y(t) = x(t)p(t). \qquad \text{(P4.28–1)}$$

(b) Suppose that $X(j\omega)$ is as depicted in Figure P4.28(a). Sketch the spectrum of $y(t)$ in eq. (P4.28–1) for each of the following choices of $p(t)$:

(i) $p(t) = \cos(t/2)$
(ii) $p(t) = \cos t$
(iii) $p(t) = \cos 2t$
(iv) $p(t) = (\sin t)(\sin 2t)$
(v) $p(t) = \cos 2t - \cos t$
(vi) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - \pi n)$
(vii) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - 2\pi n)$
(viii) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - 4\pi n)$
(ix) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - 2\pi n) - \frac{1}{2}\sum_{n=-\infty}^{+\infty} \delta(t - \pi n)$
(x) $p(t) =$ the periodic square wave shown in Figure P4.28(b).

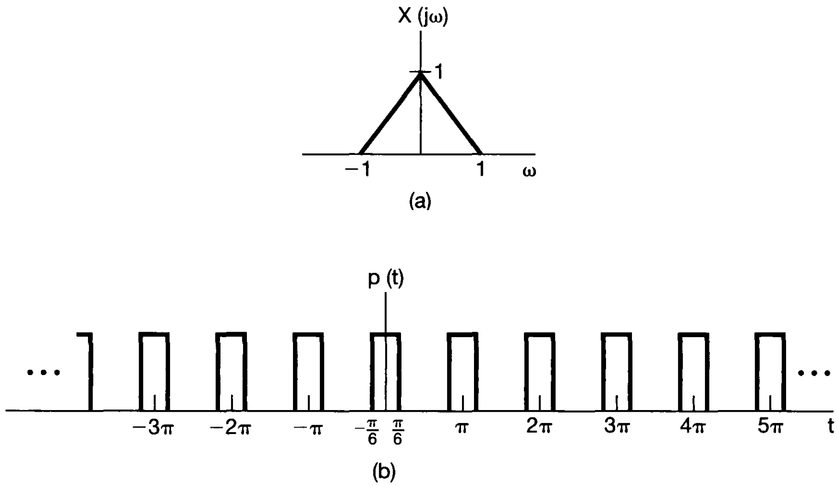

Figure P4.28

4.29. A real-valued continuous-time function $x(t)$ has a Fourier transform $X(j\omega)$ whose magnitude and phase are as illustrated in Figure P4.29(a).

The functions $x_a(t)$, $x_b(t)$, $x_c(t)$, and $x_d(t)$ have Fourier transforms whose magnitudes are identical to $X(j\omega)$, but whose phase functions differ, as shown in Figures P4.29(b)–(e). The phase functions $\sphericalangle X_a(j\omega)$ and $\sphericalangle X_b(j\omega)$ are formed by adding a linear phase to $\sphericalangle X(j\omega)$. The function $\sphericalangle X_c(j\omega)$ is formed by reflecting $\sphericalangle X(j\omega)$ about $\omega = 0$, and $\sphericalangle X_d(j\omega)$ is obtained by a combination of a reflection and an addition of a linear phase. Using the properties of Fourier transforms, determine the expressions for $x_a(t)$, $x_b(t)$, $x_c(t)$, and $x_d(t)$ in terms of $x(t)$.

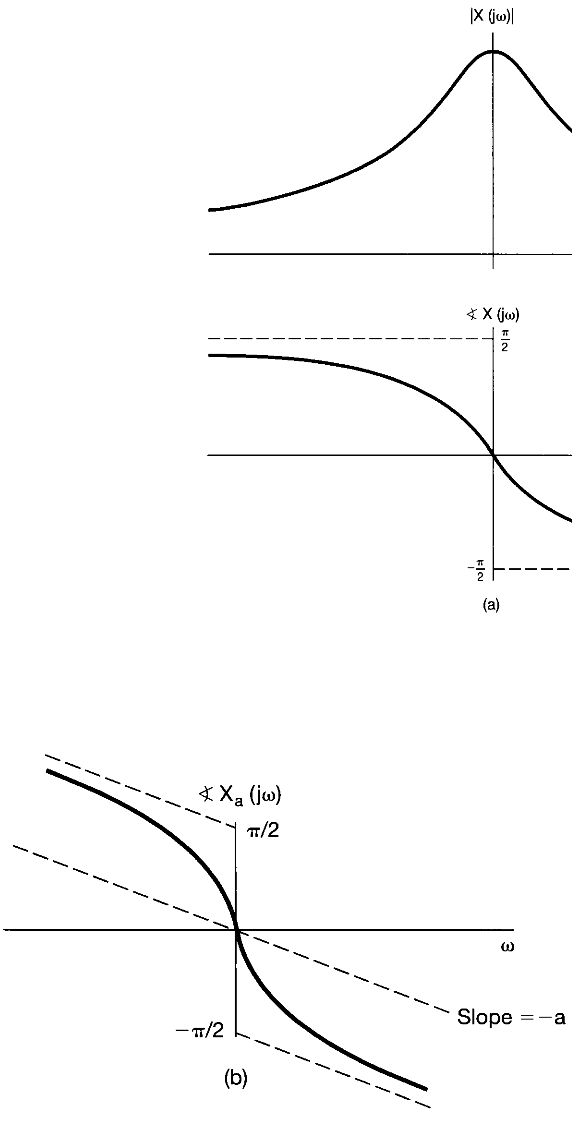

Figure P4.29

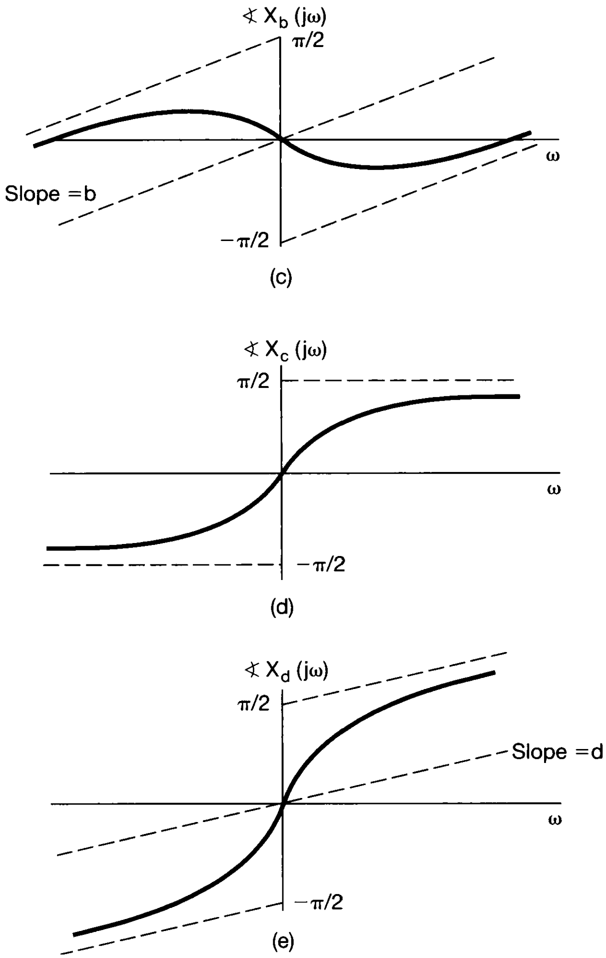

Figure P4.29 *Continued*

4.30. Suppose $g(t) = x(t)\cos t$ and the Fourier transform of the $g(t)$ is

$$G(j\omega) = \begin{cases} 1, & |\omega| \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Determine $x(t)$.

(b) Specify the Fourier transform $X_1(j\omega)$ of a signal $x_1(t)$ such that

$$g(t) = x_1(t)\cos\left(\frac{2}{3}t\right).$$

4.31. **(a)** Show that the three LTI systems with impulse responses

$$h_1(t) = u(t),$$

$$h_2(t) = -2\delta(t) + 5e^{-2t}u(t),$$

and

$$h_3(t) = 2te^{-t}u(t)$$

all have the same response to $x(t) = \cos t$.

(b) Find the impulse response of another LTI system with the same response to $\cos t$.

This problem illustrates the fact that the response to $\cos t$ cannot be used to specify an LTI system uniquely.

4.32. Consider an LTI system S with impulse response

$$h(t) = \frac{\sin(4(t-1))}{\pi(t-1)}.$$

Determine the output of S for each of the following inputs:

(a) $x_1(t) = \cos(6t + \frac{\pi}{2})$

(b) $x_2(t) = \sum_{k=0}^{\infty}(\frac{1}{2})^k \sin(3kt)$

(c) $x_3(t) = \frac{\sin(4(t+1))}{\pi(t+1)}$

(d) $x_4(t) = (\frac{\sin 2t}{\pi t})^2$

4.33. The input and the output of a stable and causal LTI system are related by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

(a) Find the impulse response of this system.

(b) What is the response of this system if $x(t) = te^{-2t}u(t)$?

(c) Repeat part (a) for the stable and causal LTI system described by the equation

$$\frac{d^2y(t)}{dt^2} + \sqrt{2}\frac{dy(t)}{dt} + y(t) = 2\frac{d^2x(t)}{dt^2} - 2x(t)$$

4.34. A causal and stable LTI system S has the frequency response

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega}.$$

(a) Determine a differential equation relating the input $x(t)$ and output $y(t)$ of S.
(b) Determine the impulse response $h(t)$ of S.
(c) What is the output of S when the input is

$$x(t) = e^{-4t}u(t) - te^{-4t}u(t)?$$

4.35. In this problem, we provide examples of the effects of nonlinear changes in phase.
(a) Consider the continuous-time LTI system with frequency response

$$H(j\omega) = \frac{a - j\omega}{a + j\omega},$$

where $a > 0$. What is the magnitude of $H(j\omega)$? What is $\sphericalangle H(j\omega)$? What is the impulse response of this system?
(b) Determine the output of the system of part (a) with $a = 1$ when the input is

$$\cos(t/\sqrt{3}) + \cos t + \cos \sqrt{3}t.$$

Roughly sketch both the input and the output.

4.36. Consider an LTI system whose response to the input

$$x(t) = [e^{-t} + e^{-3t}]u(t)$$

is

$$y(t) = [2e^{-t} - 2e^{-4t}]u(t).$$

(a) Find the frequency response of this system.
(b) Determine the system's impulse response.
(c) Find the differential equation relating the input and the output of this system.

ADVANCED PROBLEMS

4.37. Consider the signal $x(t)$ in Figure P4.37.
(a) Find the Fourier transform $X(j\omega)$ of $x(t)$.
(b) Sketch the signal

$$\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t - 4k).$$

(c) Find another signal $g(t)$ such that $g(t)$ is not the same as $x(t)$ and

$$\tilde{x}(t) = g(t) * \sum_{k=-\infty}^{\infty} \delta(t - 4k).$$

(d) Argue that, although $G(j\omega)$ is different from $X(j\omega)$, $G(j\frac{\pi k}{2}) = X(j\frac{\pi k}{2})$ for all integers k. You should not explicitly evaluate $G(j\omega)$ to answer this question.

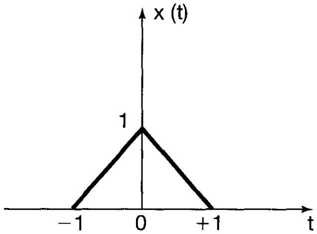

Figure P4.37

4.38. Let $x(t)$ be any signal with Fourier transform $X(j\omega)$. The frequency-shift property of the Fourier transform may be stated as

$$e^{j\omega_0 t}x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0)).$$

(a) Prove the frequency-shift property by applying the frequency shift to the analysis equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

(b) Prove the frequency-shift property by utilizing the Fourier transform of $e^{j\omega_0 t}$ in conjunction with the multiplication property of the Fourier transform.

4.39. Suppose that a signal $x(t)$ has Fourier transform $X(j\omega)$. Now consider another signal $g(t)$ whose shape is the same as the shape of $X(j\omega)$; that is,

$$g(t) = X(jt).$$

(a) Show that the Fourier transform $G(j\omega)$ of $g(t)$ has the same shape as $2\pi x(-t)$; that is, show that

$$G(j\omega) = 2\pi x(-\omega).$$

(b) Using the fact that

$$\mathcal{F}\{\delta(t + B)\} = e^{jB\omega}$$

in conjunction with the result from part (a), show that

$$\mathcal{F}\{e^{jBt}\} = 2\pi\,\delta(\omega - B).$$

4.40. Use properties of the Fourier transform to show by induction that the Fourier transform of

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t),\ a > 0,$$

is

$$\frac{1}{(a + j\omega)^n}.$$

4.41. In this problem, we derive the multiplication property of the continuous-time Fourier transform. Let $x(t)$ and $y(t)$ be two continuous-time signals with Fourier transforms $X(j\omega)$ and $Y(j\omega)$, respectively. Also, let $g(t)$ denote the inverse Fourier transform of $\frac{1}{2\pi}\{X(j\omega) * Y(j\omega)\}$.

(a) Show that

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j(\omega - \theta))e^{j\omega t}\,d\omega\right]d\theta.$$

(b) Show that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j(\omega - \theta))e^{j\omega t}\,d\omega = e^{j\theta t}y(t).$$

(c) Combine the results of parts (a) and (b) to conclude that

$$g(t) = x(t)y(t).$$

4.42. Let

$$g_1(t) = \{[\cos(\omega_0 t)]x(t)\} * h(t) \qquad \text{and} \qquad g_2(t) = \{[\sin(\omega_0 t)]x(t)\} * h(t),$$

where

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk100t}$$

is a real-valued periodic signal and $h(t)$ is the impulse response of a stable LTI system.

(a) Specify a value for ω_0 and any necessary constraints on $H(j\omega)$ to ensure that

$$g_1(t) = \mathcal{R}e\{a_5\} \qquad \text{and} \qquad g_2(t) = \mathcal{I}m\{a_5\}.$$

(b) Give an example of $h(t)$ such that $H(j\omega)$ satisfies the constraints you specified in part (a).

4.43. Let

$$g(t) = x(t)\cos^2 t * \frac{\sin t}{\pi t}.$$

Assuming that $x(t)$ is real and $X(j\omega) = 0$ for $|\omega| \geq 1$, show that there exists an LTI system S such that

$$x(t) \xrightarrow{S} g(t).$$

4.44. The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the equation

$$\frac{dy(t)}{dt} + 10y(t) = \int_{-\infty}^{+\infty} x(\tau)z(t-\tau)\,d\tau - x(t),$$

where $z(t) = e^{-t}u(t) + 3\,\delta(t)$.

(a) Find the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$ of this system.

(b) Determine the impulse response of the system.

4.45. In the discussion in Section 4.3.7 of Parseval's relation for continuous-time signals, we saw that

$$\int_{-\infty}^{+\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\,d\omega.$$

This says that the total energy of the signal can be obtained by integrating $|X(j\omega)|^2$ over all frequencies. Now consider a real-valued signal $x(t)$ processed by the ideal bandpass filter $H(j\omega)$ shown in Figure P4.45. Express the energy in the output signal $y(t)$ as an integration over frequency of $|X(j\omega)|^2$. For Δ sufficiently small so that $|X(j\omega)|$ is approximately constant over a frequency interval of width Δ, show that the energy in the output $y(t)$ of the bandpass filter is approximately proportional to $\Delta|X(j\omega_0)|^2$.

On the basis of the foregoing result, $\Delta|X(j\omega_0)|^2$ is proportional to the energy in the signal in a bandwidth Δ around the frequency ω_0. For this reason, $|X(j\omega)|^2$ is often referred to as the *energy-density spectrum* of the signal $x(t)$.

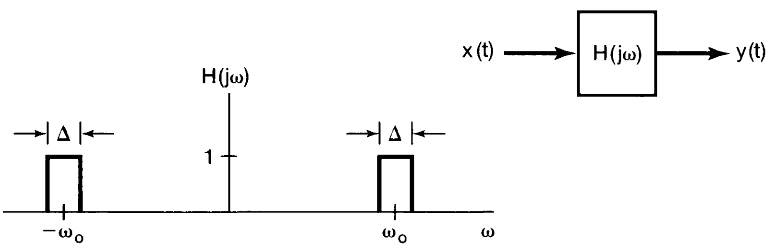

Figure P4.45

4.46. In Section 4.5.1, we discussed the use of amplitude modulation with a complex exponential carrier to implement a bandpass filter. The specific system was shown in Figure 4.26, and if only the real part of $f(t)$ is retained, the equivalent bandpass filter is that shown in Figure 4.30.

In Figure P4.46, we indicate an implementation of a bandpass filter using sinusoidal modulation and lowpass filters. Show that the output $y(t)$ of the system is identical to that which would be obtained by retaining only $\mathcal{R}e\{f(t)\}$ in Figure 4.26.

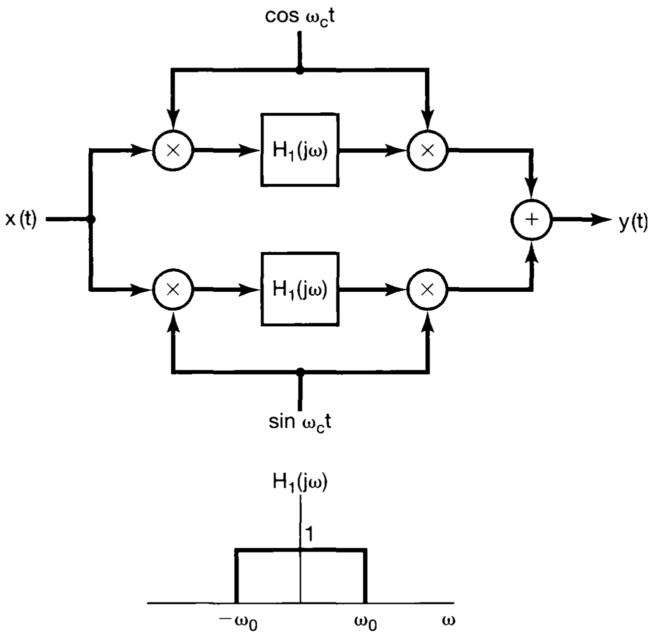

Figure P4.46

4.47. An important property of the frequency response $H(j\omega)$ of a continuous-time LTI system with a real, causal impulse response $h(t)$ is that $H(j\omega)$ is completely specified by its real part, $\mathcal{R}e\{H(j\omega)\}$. The current problem is concerned with deriving and examining some of the implications of this property, which is generally referred to as *real-part sufficiency*.

(a) Prove the property of real-part sufficiency by examining the signal $h_e(t)$, which is the even part of $h(t)$. What is the Fourier transform of $h_e(t)$? Indicate how $h(t)$ can be recovered from $h_e(t)$.

(b) If the real part of the frequency response of a causal system is

$$\mathcal{R}e\{H(j\omega)\} = \cos\omega,$$

what is $h(t)$?

(c) Show that $h(t)$ can be recovered from $h_o(t)$, the odd part of $h(t)$, for every value of t except $t = 0$. Note that if $h(t)$ does not contain any singularities [$\delta(t)$, $u_1(t)$, $u_2(t)$, etc.] at $t = 0$, then the frequency response

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}\,dt$$

will not change if $h(t)$ is set to some arbitrary finite value at the single point $t = 0$. Thus, in this case, show that $H(j\omega)$ is also completely specified by its imaginary part.

EXTENSION PROBLEMS

4.48. Let us consider a system with a real and causal impulse response $h(t)$ that does not have any singularities at $t = 0$. In Problem 4.47, we saw that either the real or the imaginary part of $H(j\omega)$ completely determines $H(j\omega)$. In this problem we derive an explicit relationship between $H_R(j\omega)$ and $H_I(j\omega)$, the real and imaginary parts of $H(j\omega)$.

(a) To begin, note that since $h(t)$ is causal,

$$h(t) = h(t)u(t), \qquad \text{(P4.48–1)}$$

except perhaps at $t = 0$. Now, since $h(t)$ contains no singularities at $t = 0$, the Fourier transforms of both sides of eq. (P4.48–1) must be identical. Use this fact, together with the multiplication property, to show that

$$H(j\omega) = \frac{1}{j\pi}\int_{-\infty}^{+\infty} \frac{H(j\eta)}{\omega - \eta} d\eta. \qquad \text{(P4.48–2)}$$

Use eq. (P4.48–2) to determine an expression for $H_R(j\omega)$ in terms of $H_I(j\omega)$ and one for $H_I(j\omega)$ in terms of $H_R(j\omega)$.

(b) The operation

$$y(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{x(\tau)}{t - \tau} d\tau \qquad \text{(P4.48–3)}$$

is called the *Hilbert transform.* We have just seen that the real and imaginary parts of the transform of a real, causal impulse response $h(t)$ can be determined from one another using the Hilbert transform.

Now consider eq. (P4.48–3), and regard $y(t)$ as the output of an LTI system with input $x(t)$. Show that the frequency response of this system is

$$H(j\omega) = \begin{cases} -j, & \omega > 0 \\ j, & \omega < 0 \end{cases}.$$

(c) What is the Hilbert transform of the signal $x(t) = \cos 3t$?

4.49. Let $H(j\omega)$ be the frequency response of a continuous-time LTI system, and suppose that $H(j\omega)$ is real, even, and positive. Also, assume that

$$\max_{\omega}\{H(j\omega)\} = H(0).$$

(a) Show that:

(i) The impulse response, $h(t)$, is real.

(ii) $\max\{|h(t)|\} = h(0)$.

Hint: If $f(t, \omega)$ is a complex function of two variables, then

$$\left|\int_{-\infty}^{+\infty} f(t,\omega)\, d\omega\right| \le \int_{-\infty}^{+\infty} |f(t,\omega)|\, d\omega.$$

(b) One important concept in system analysis is the *bandwidth* of an LTI system. There are many different mathematical ways in which to define bandwidth, but they are related to the qualitative and intuitive idea that a system with frequency response $G(j\omega)$ essentially "stops" signals of the form $e^{j\omega t}$ for values of ω where $G(j\omega)$ vanishes or is small and "passes" those complex exponentials in the band of frequency where $G(j\omega)$ is not small. The width of this band is the bandwidth. These ideas will be made much clearer in Chapter 6, but for now we will consider a special definition of bandwidth for those systems with frequency responses that have the properties specified previously for $H(j\omega)$. Specifically, one definition of the bandwidth B_w of such a system is the width of the rectangle of height $H(j0)$ that has an area equal to the area under $H(j\omega)$. This is illustrated in Figure P4.49(a). Note that since $H(j0) = \max_\omega H(j\omega)$, the frequencies within the band indicated in the figure are those for which $H(j\omega)$ is largest. The exact choice of the width in the figure is, of course, a bit arbitrary, but we have chosen one definition that allows us to compare different systems and to make precise a very important relationship between time and frequency.

What is the bandwidth of the system with frequency response

$$H(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}?$$

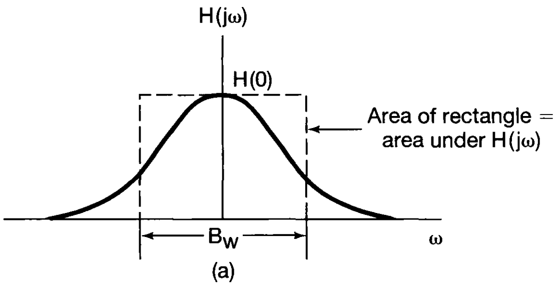

Figure P4.49a

(c) Find an expression for the bandwidth B_w in terms of $H(j\omega)$.

(d) Let $s(t)$ denote the step response of the system set out in part (a). An important measure of the speed of response of a system is the *rise time,* which, like the bandwidth, has a qualitative definition, leading to many possible mathematical definitions, one of which we will use. Intuitively, the rise time of a system is a measure of how fast the step response rises from zero to its final value,

$$s(\infty) = \lim_{t\to\infty} s(t).$$

Thus, the smaller the rise time, the faster is the response of the system. For the system under consideration in this problem, we will define the rise time as

$$t_r = \frac{s(\infty)}{h(0)}.$$

Since

$$s'(t) = h(t),$$

and also because of the property that $h(0) = \max_t h(t)$, t_r is the time it would take to go from zero to $s(\infty)$ while maintaining the maximum rate of change of $s(t)$. This is illustrated in Figure P4.49(b).

Find an expression for t_r in terms of $H(j\omega)$.

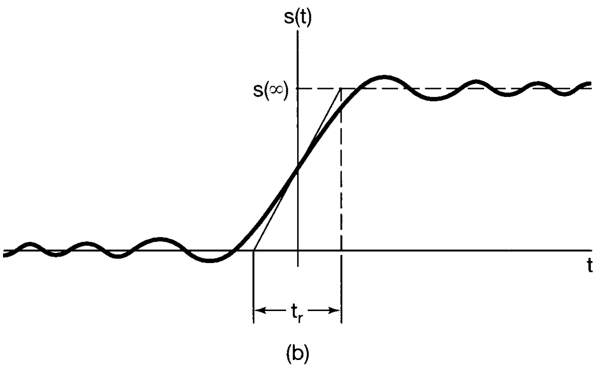

Figure P4.49b

(e) Combine the results of parts (c) and (d) to show that

$$B_w t_r = 2\pi. \qquad \text{(P4.49–1)}$$

Thus, we *cannot* independently specify both the rise time and the bandwidth of our system. For example, eq. (P4.49–1) implies that, if we want a fast system (t_r small), the system must have a large bandwidth. This is a fundamental trade-off that is of central importance in many problems of system design.

4.50. In Problems 1.45 and 2.67, we defined and examined several of the properties and uses of correlation functions. In the current problem, we examine the properties of such functions in the frequency domain. Let $x(t)$ and $y(t)$ be two real signals. Then the cross-correlation function of $x(t)$ and $y(t)$ is defined as

$$\phi_{xy}(t) = \int_{-\infty}^{+\infty} x(t+\tau)y(\tau)\,d\tau.$$

Similarly, we can define $\phi_{yx}(t)$, $\phi_{xx}(t)$, and $\phi_{yy}(t)$. [The last two of these are called the autocorrelation functions of the signals $x(t)$ and $y(t)$, respectively.] Let $\Phi_{xy}(j\omega)$, $\Phi_{yx}(j\omega)$, $\Phi_{xx}(j\omega)$, and $\Phi_{yy}(j\omega)$ denote the Fourier transforms of $\phi_{xy}(t)$, $\phi_{yx}(t)$, $\phi_{xx}(t)$, and $\phi_{yy}(t)$, respectively.

(a) What is the relationship between $\Phi_{xy}(j\omega)$ and $\Phi_{yx}(j\omega)$?

(b) Find an expression for $\Phi_{xy}(j\omega)$ in terms of $X(j\omega)$ and $Y(j\omega)$.

(c) Show that $\Phi_{xx}(j\omega)$ is real and nonnegative for every ω.

(d) Suppose now that $x(t)$ is the input to an LTI system with a real-valued impulse response and with frequency response $H(j\omega)$ and that $y(t)$ is the output. Find expressions for $\Phi_{xy}(j\omega)$ and $\Phi_{yy}(j\omega)$ in terms of $\Phi_{xx}(j\omega)$ and $H(j\omega)$.

(e) Let $x(t)$ be as is illustrated in Figure P4.50, and let the LTI system impulse response be $h(t) = e^{-at}u(t)$, $a > 0$. Compute $\Phi_{xx}(j\omega)$, $\Phi_{xy}(j\omega)$, and $\Phi_{yy}(j\omega)$ using the results of parts (a)–(d).

(f) Suppose that we are given the following Fourier transform of a function $\phi(t)$:

$$\Phi(j\omega) = \frac{\omega^2 + 100}{\omega^2 + 25}.$$

Find the impulse responses of *two* causal, stable LTI systems that have autocorrelation functions equal to $\phi(t)$. Which one of these has a causal, stable inverse?

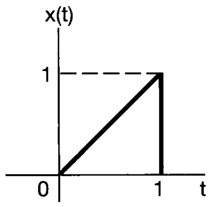

Figure P4.50

4.51. (a) Consider two LTI systems with impulse responses $h(t)$ and $g(t)$, respectively, and suppose that these systems are inverses of one another. Suppose also that the systems have frequency responses denoted by $H(j\omega)$ and $G(j\omega)$, respectively. What is the relationship between $H(j\omega)$ and $G(j\omega)$?

(b) Consider the continuous-time LTI system with frequency response

$$H(j\omega) = \begin{cases} 1, & 2 < |\omega| < 3 \\ 0, & \text{otherwise} \end{cases}.$$

(i) Is it possible to find an input $x(t)$ to this system such that the output is as depicted in Figure P4.50? If so, find $x(t)$. If not, explain why not.

(ii) Is this system invertible? Explain your answer.

(c) Consider an auditorium with an echo problem. As discussed in Problem 2.64, we can model the acoustics of the auditorium as an LTI system with an impulse response consisting of an impulse train, with the kth impulse in the train corresponding to the kth echo. Suppose that in this particular case the impulse response is

$$h(t) = \sum_{k=0}^{\infty} e^{-kT}\,\delta(t - kT),$$

where the factor e^{-kT} represents the attenuation of the kth echo.

In order to make a high-quality recording from the stage, the effect of the echoes must be removed by performing some processing of the sounds sensed by the recording equipment. In Problem 2.64, we used convolutional techniques to consider one example of the design of such a processor (for a different acoustic model). In the current problem, we will use frequency-domain techniques. Specifically, let $G(j\omega)$ denote the frequency response of the LTI system to be used to process the sensed acoustic signal. Choose $G(j\omega)$ so that the echoes are completely removed and the resulting signal is a faithful reproduction of the original stage sounds.

(d) Find the differential equation for the inverse of the system with impulse response

$$h(t) = 2\,\delta(t) + u_1(t).$$

(e) Consider the LTI system initially at rest and described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 9y(t) = \frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t).$$

The inverse of this system is also initially at rest and described by a differential equation. Find the differential equation describing the inverse, and find the impulse responses $h(t)$ and $g(t)$ of the original system and its inverse.

4.52. Inverse systems frequently find application in problems involving imperfect measuring devices. For example, consider a device for measuring the temperature of a liquid. It is often reasonable to model such a device as an LTI system that, because of the response characteristics of the measuring element (e.g., the mercury in a thermometer), does not respond instantaneously to temperature changes. In particular, assume that the response of this device to a unit step in temperature is

$$s(t) = (1 - e^{-t/2})u(t). \qquad \text{(P4.52–1)}$$

(a) Design a compensatory system that, when provided with the output of the measuring device, produces an output equal to the instantaneous temperature of the liquid.

(b) One of the problems that often arises in using inverse systems as compensators for measuring devices is that gross inaccuracies in the indicated temperature may occur if the actual output of the measuring device produces errors due to small, erratic phenomena in the device. Since there always are such sources of error in real systems, one must take them into account. To illustrate this, consider a measuring device whose overall output can be modeled as the sum of the response of the measuring device characterized by eq. (P4.52–1) and an interfering "noise" signal $n(t)$. Such a model is depicted in Figure P4.52(a), where we have also included the inverse system of part (a), which now has as its input the *overall* output of the measuring device. Suppose that $n(t) = \sin \omega t$. What is the contribution of $n(t)$ to the output of the inverse system, and how does this output change as ω is increased?

(c) The issue raised in part (b) is an important one in many applications of LTI system analysis. Specifically, we are confronted with the fundamental tradeoff between the speed of response of the system and the ability of the system to attenuate high-frequency interference. In part (b) we saw that this tradeoff implied that, by attempting to speed up the response of a measuring device (by means of an inverse system), we produced a system that would also amplify

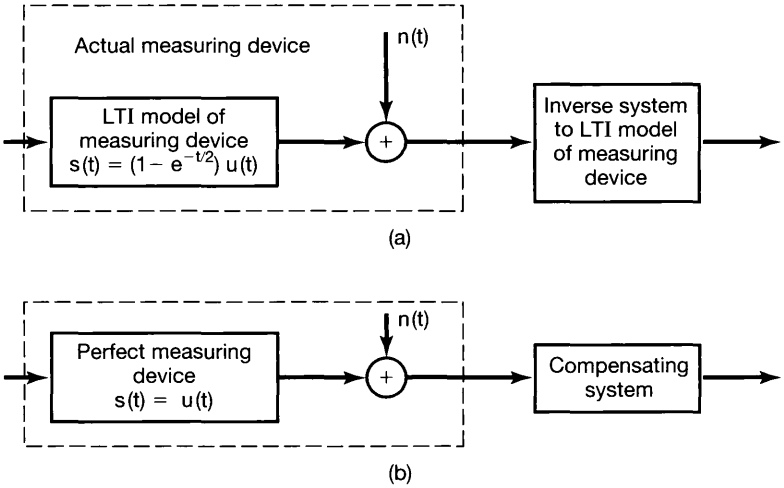

Figure P4.52

corrupting sinusoidal signals. To illustrate this concept further, consider a measuring device that responds instantaneously to changes in temperature, but that also is corrupted by noise. The response of such a system can be modeled, as depicted in Figure P4.52(b), as the sum of the response of a perfect measuring device and a corrupting signal $n(t)$. Suppose that we wish to design a compensatory system that will *slow down* the response to actual temperature variations, but also will attenuate the noise $n(t)$. Let the impulse response of this system be

$$h(t) = ae^{-at}u(t).$$

Choose a so that the overall system of Figure P4.52(b) responds as quickly as possible to a step change in temperature, subject to the constraint that the amplitude of the portion of the output due to the noise $n(t) = \sin 6t$ is no larger than 1/4.

4.53. As mentioned in the text, the techniques of Fourier analysis can be extended to signals having two independent variables. As their one-dimensional counterparts do in some applications, these techniques play an important role in other applications, such as image processing. In this problem, we introduce some of the elementary ideas of two-dimensional Fourier analysis.

Let $x(t_1, t_2)$ be a signal that depends upon two independent variables t_1 and t_2. The *two-dimensional Fourier transform* of $x(t_1, t_2)$ is defined as

$$X(j\omega_1, j\omega_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(t_1, t_2)e^{-j(\omega_1 t_1 + \omega_2 t_2)}\,dt_1\,dt_2.$$

(a) Show that this double integral can be performed as two successive one-dimensional Fourier transforms, first in t_1 with t_2 regarded as fixed and then in t_2.

(b) Use the result of part (a) to determine the inverse transform—that is, an expression for $x(t_1, t_2)$ in terms of $X(j\omega_1, j\omega_2)$.

(c) Determine the two-dimensional Fourier transforms of the following signals:

(i) $x(t_1, t_2) = e^{-t_1+2t_2}u(t_1 - 1)u(2 - t_2)$

(ii) $x(t_1, t_2) = \begin{cases} e^{-|t_1|-|t_2|}, & \text{if } -1 < t_1 \leq 1 \text{ and } -1 \leq t_2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(iii) $x(t_1, t_2) = \begin{cases} e^{-|t_1|-|t_2|}, & \text{if } 0 \leq t_1 \leq 1 \text{ or } 0 \leq t_2 \leq 1 \text{ (or both)} \\ 0, & \text{otherwise} \end{cases}$

(iv) $x(t_1, t_2)$ as depicted in Figure P4.53.

(v) $e^{-|t_1+t_2|-|t_1-t_2|}$

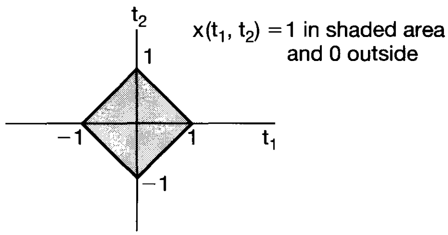

Figure P4.53

(d) Determine the signal $x(t_1, t_2)$ whose two-dimensional Fourier transform is

$$X(j\omega_1, j\omega_2) = \frac{2\pi}{4 + j\omega_1}\delta(\omega_2 - 2\omega_1).$$

(e) Let $x(t_1, t_2)$ and $h(t_1, t_2)$ be two signals with two-dimensional Fourier transforms $X(j\omega_1, j\omega_2)$ and $H(j\omega_1, j\omega_2)$, respectively. Determine the transforms of the following signals in terms of $X(j\omega_1, j\omega_2)$ and $H(j\omega_1, j\omega_2)$:

(i) $x(t_1 - T_1, t_2 - T_2)$

(ii) $x(at_1, bt_2)$

(iii) $y(t_1, t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau_1, \tau_2)h(t_1 - \tau_1, t_2 - \tau_2)\,d\tau_1\,d\tau_2$