$H(s)$ = transfer function of the system   on final!

2. Response of LTI systems to $\mathbb{C}$ exponentials

eigen function

CT

$s = 3 - 7j$ a number

$s \in \mathbb{C}$

$$e^{st} \longrightarrow \boxed{LTI} \longrightarrow \overbrace{H(s)}^{\text{number}} e^{st}$$

complex exponential signal

$\uparrow$ transfer fct

$\omega \in \mathbb{R}$

$$e^{j\omega t} \longrightarrow \boxed{LTI} \longrightarrow \overbrace{\mathcal{H}(\omega)}^{\text{num - depends on } \omega} e^{j\omega t}$$

$\uparrow$ frequency response

DT

$z \in \mathbb{C}$

$$z^n \longrightarrow \boxed{LTI} \longrightarrow H(z)\, z^n$$

$\uparrow$ transfer fct.

$\omega \in \mathbb{R}$

$$e^{j\omega n} \longrightarrow \boxed{LTI} \longrightarrow \mathcal{H}(\omega) e^{j\omega n}$$

$\uparrow$ frequency response.

Why?

In CT, let $x(t) = e^{st}$

$$
\begin{aligned}
\text{then } y(t) &= x(t) * h(t) \\
&= h(t) * x(t) \\
&= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau \\
&= \underbrace{e^{st}}_{x(t)} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{H(s)}
\end{aligned}
$$

Special case $s = j\omega$

$$\text{then } y(t) = e^{j\omega t} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau}_{\mathcal{H}(\omega)}$$

Examples $x(t) = e^{t}$, here $s=1$

$$\underset{e^{t}}{x(t)} \longrightarrow \boxed{\text{LTI } h(t)} \longrightarrow e^{t} H(1)$$

$$y(t) = x(t) * h(t)$$
$$= h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} h(\tau)\, e^{t-\tau}\, d\tau = e^{t} \int_{-\infty}^{\infty} h(\tau)\, e^{-\tau}\, d\tau$$

$$\underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-\tau}\, d\tau}_{\text{number}} = H(1)$$

eigenvalue of the eigenfunction e^{t}

$x(t) = e^{j2\pi f_0 t}$

$$y(t) = h(t) * x(t)$$
$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{\infty} h(\tau)\, e^{j2\pi f_0(t-\tau)}\, d\tau$$
$$= e^{j2\pi f_0 t} \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f_0 \tau}\, d\tau$$

Complex num,

freq

number, $\underset{\text{transfer function}}{H}(j2\pi f_0) = \underset{\text{freq response}}{H}(2\pi f_0)$

Why?

In DT, let $x[n] = z^n$

$$
\begin{aligned}
\text{then } y[n] &= x[n] * h[n] \\
&= h[n] * x[n] \\
&= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k} \\
&= z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}}_{H(z)}
\end{aligned}
$$

Special case $z = e^{j\omega}$

$$
\text{then } y[n] = e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}}_{H(\omega)}
$$

Examples: Let $x[n] = j^n$

$$
\begin{aligned}
\text{then } y[n] &= x[n] * h[n] \\
&= h[n] * x[n] \\
&= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]\, j^{n-k} \\
&= j^n \underbrace{\sum_{k=-\infty}^{\infty} h[k]\, j^{-k}}_{\text{a number}}
\end{aligned}
$$

$$
j^n \longrightarrow \boxed{} \longrightarrow \underbrace{j^n}_{\text{eigenfunction}} \cdot \overbrace{\text{number}}^{\text{eigenvalue}}
$$

3. Fourier series of CT periodic Signals

Let $x(t)$ be a periodic signal with period T.
Write $\omega_0 = 2\pi / T$.
Then the Fourier series of $x(t)$ is

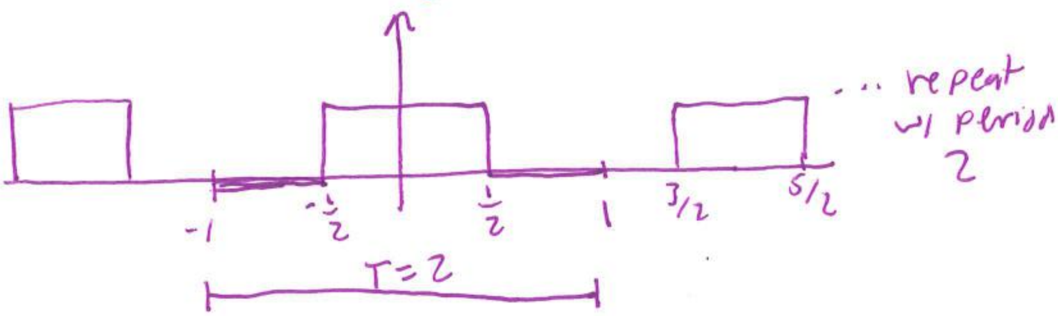
$$x(t) = \sum_{k=-\infty}^{\infty} a_k \underbrace{e^{jk\omega_0 t}}_{\text{harmonic family all share period } T} \quad \text{where } \overset{\text{coefficients}}{a_k} = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t}\, dt$$

- dot product
- if complex
  ↳ conjugate

Projection of $x(t)$ onto $e^{jk\omega_0 t}$

1 period

Examples:



$$\left\{ e^{jk\frac{2\pi}{2}t} \right\}_{k\in\mathbb{Z}} = \left\{ e^{jk\pi t} \right\}$$

$k=0$ $\quad a_0 = \frac{1}{2}\int_{-1}^{1} x(t) e^{j0\pi t}\, dt$

$$= \frac{1}{2}\int_{-1/2}^{1/2} 1\, dt = \boxed{\frac{1}{2}}$$ average signal over 1 period

$$a_k = \frac{1}{2}\int_{-1}^{1} x(t) e^{-jk\pi t}\, dt$$

so

$$x(t) = \frac{1}{2} + \sum_{k\neq 0} \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} e^{jk\pi t}$$

$$= \frac{1}{2}\int_{-1/2}^{1/2} e^{-jk\pi t}\, dt$$

$$= \frac{1}{2} \left. \frac{e^{-jk\pi t}}{-jk\pi} \right|_{-1/2}^{1/2} = \frac{1}{2}\left[\frac{e^{-jk\frac{\pi}{2}} - e^{jk\frac{\pi}{2}}}{-j\pi k} \right] = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$

Ex $x(t) = 3\cos 3t + (1+6j)\sin 6t$

$$= 3\left(\frac{e^{3jt}+e^{-3jt}}{2}\right) + (1+6j)\left(\frac{e^{6jt}-e^{-6jt}}{2j}\right)$$

$$= \frac{3}{2}e^{3jt} + \frac{3}{2}e^{-3jt} + \left(\frac{1+6j}{2j}\right)e^{6jt} - \left(\frac{1+6j}{2j}\right)e^{-6jt} \quad (*)$$

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, take $\omega_0 = 3 \rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk3t}$

$(**) \; = \ldots \underbrace{a_{-2}e^{-6jt}}_{k=-2} + \underbrace{a_{-1}e^{-3jt}}_{k=-1} + a_0 + a_1 e^{3jt} + a_2 e^{6jt}$

a_0 not in $(*)$ so $=0$

by comparison of $(*)$ and $(**)$

$a_1 = \frac{3}{2} \quad a_2 = \frac{1+6j}{2j}$

$a_{-1} = \frac{3}{2} \quad a_{-2} = \frac{-(1+6j)}{2j}$

$a_0 = 0$

EX] DTFT of periodic signal

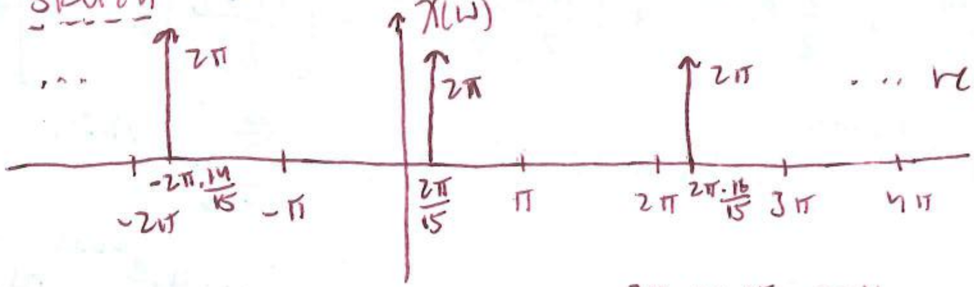
* helps w/ HW

① $x[n] = e^{j\frac{2\pi}{15}n}$, period $N=15$

guess is $X(\omega) = 2\pi\,\delta(\omega - \frac{2\pi}{15})$ <u>if</u> $0 \le \omega < 2\pi$

$X(\omega + 2\pi) = X(\omega),\ \forall \omega$ (repeats)

<u>sketch</u>



$\frac{2\pi}{15} - 2\pi = -2\pi\cdot\frac{14}{15}$

$\frac{2\pi}{15} + \frac{2\pi}{1}\cdot\frac{15}{15} = 2\pi\frac{16}{15}$

$$\mathcal{F}^{-1}(X(\omega)) = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_0^{2\pi} 2\pi\,\delta(\omega - \tfrac{2\pi}{15})e^{j\omega n}\,d\omega$$

$$= e^{j\frac{2\pi}{15}n}$$

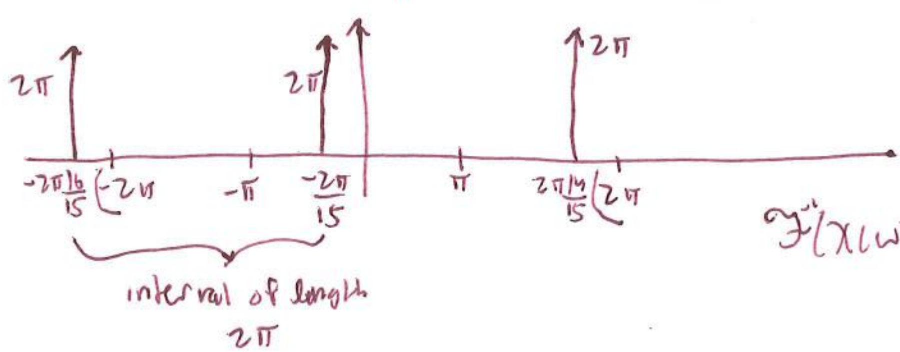
$$= x[n]\ \checkmark$$

* $\forall$ "for all"
* $\exists$ "there exists"

② $x[n] = e^{-j\frac{2\pi}{15}n}$ period $N=15$

guess is $X(\omega) = 2\pi\,\delta(\omega + \frac{2\pi}{15})$, for $-2\pi < \omega \le 0$

$X(\omega + 2\pi) = X(\omega),\ \forall \omega$



$$\mathcal{F}^{-1}(X(\omega)) = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-2\pi}^{0} X(\omega)e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-2\pi}^{0} 2\pi\,\delta(\omega + \tfrac{2\pi}{15})e^{j\omega n}\,d\omega$$

$$= e^{-j\frac{2\pi}{15}n}$$

$$= x[n]\ \checkmark$$

4. Fourier Series of DT periodic Signals

Let $x[n]$ be a periodic signal with period N.
Then the Fourier series of $x[n]$ is

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n} \quad \text{where} \quad a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n}.$$

dot prod

n_0+N-1

one period

$n=n_0$

Why is the sum finite?

Because the set $\left\{ e^{jk\frac{2\pi}{N}n} \right\}_{k\in\mathbb{Z}}$ of harmonically related exponentials only contains a finite number of distinct DT functions.

periodic DT signal

periodic DT signal

$[x[0], x[1], x[2] \ldots x[n-1]] \longrightarrow [a_0, a_1, a_2 \ldots a_{N-1}]$

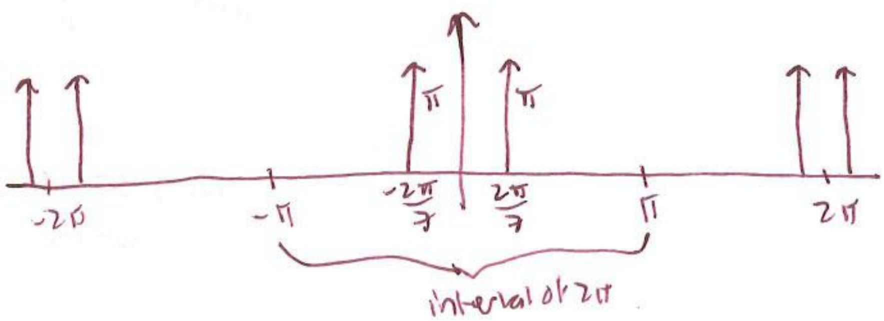
$a_N = a_0$

$a_{N+1} = a_1$

(3) $x[n] = \cos(\frac{2\pi}{7}n) \qquad N=7$

$$= \frac{e^{j\frac{2\pi}{7}n} + e^{-j\frac{2\pi}{7}n}}{2}$$

guess $\rightarrow X(\omega) = \pi\delta(\omega - \frac{2\pi}{7}) + \pi\delta(\omega + \frac{2\pi}{7})$, for $-\pi \le \omega < \pi$



$$\mathcal{F}^{-1}(X(\omega)) = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\pi\delta(\omega - \tfrac{2\pi}{7}) + \pi\delta(\omega + \tfrac{2\pi}{7})\right)e^{j\omega n}\,d\omega$$

$$= \frac{e^{j\frac{2\pi}{7}n} + e^{-j\frac{2\pi}{7}n}}{2} = \cos(\tfrac{2\pi}{7}n)$$

~~*~~ Kchov! tricks too! Could specify on tests e^{jt}

Examples: $x[n] = \sin\left(3\pi n + \frac{\pi}{2}\right) = (-1)^n$

$$\underline{\underline{N}} = \frac{2\pi}{3\pi} K \quad K=3 \rightarrow N=2$$

integer

$$x[0] = \sin\left(\frac{\pi}{2}\right) = 1$$
$$x[1] = \sin\left(3\pi + \frac{\pi}{2}\right)$$
$$x[2] = 1$$
$$x[3] = -1$$

compute fourier coefficent

$$a_0 = \frac{1}{2}\sum_{n=0}^{1} x[n] e^0 = \frac{1}{2}\sum_{n=0}^{1} x[n] \quad \text{(avg } x[n] \text{ over 1 period)}$$

$$= 0$$

$$a_1 = \frac{1}{2}\sum_{n=0}^{1} x[n] e^{-j\frac{2\pi}{2}n} = \frac{1}{2}\sum_{n=0}^{1} x[n] e^{-j\pi n}$$

$$= \frac{1}{2} x[0] e^0 + \frac{1}{2} x[1] e^{-j\pi}$$

$$= \frac{1}{2}(1)(1) + \frac{1}{2}(-1)(-1) = 1$$

Ans
$a_0 = 0$
$a_1 = 1$
$a_{k+2} = a_k$, for all other K
(repeat periodically w/ period 2)

Quick method

$$x[n] = (-1)^n = (e^{j\pi})^n = e^{j\pi n} + \cancel{0} + \cancel{0}\cancel{e^{j3\pi n}}$$

compare w/ Fourier series formula

$$x[n] = \sum_{k=0}^{1} a_k e^{jk\pi n} = a_0 e^0 + a_1 e^{j\pi n} = \boxed{a_0 + a_1 e^{j\pi n}}$$

$k=0,\ k=1$

$\Rightarrow a_0 = 0$
$a_1 = 1$
$a_2 = 0$
$a_3 = 1$
$a_4 = 0$
$a_5 = 1$

$$a_k = \begin{cases} 0, & \text{if } k \text{ even} \\ 1, & \text{if } k \text{ odd} \end{cases}$$

freq domain

$$a_k \longmapsto a_k H(k\omega_0)$$

time domain

$$\sum a_k e^{jk\frac{2\pi}{T}t} \longmapsto \sum a_k H(k\omega_0) e^{jk\frac{2\pi}{T}t}$$

5. Fourier Series and LTI systems

Recall $\underset{\text{periodic}}{e^{j\omega t}} \longrightarrow \boxed{LTI} \longrightarrow e^{j\omega t} H(\omega)$

$$\Rightarrow \quad x(t) = \sum_k a_k e^{jk\omega_0 t} \longrightarrow \boxed{LTI} \longrightarrow \sum_k a_k e^{jk\omega_0 t} \underbrace{H(k\omega_0)}_{\text{freq. responce (mag)}}$$

Example: if $x(t) = \sin(\underbrace{2\pi 440 t}_{T=\frac{1}{440}}) + \sin(\underbrace{2\pi 880 t}_{T=\frac{1}{880} \Rightarrow 2T = \frac{1}{440}})$

$$\sum_{k=-\infty}^{\infty} a_k e^{jk2\pi 440 t} \underset{\text{compare}}{=} x(t) \quad \begin{cases} = \underbrace{\frac{1}{2j}}_{a_1} e^{j2\pi 440 t} - \underbrace{\frac{1}{2j}}_{a_{-1}} e^{-j2\pi 440 t} \\ + \underbrace{\frac{1}{2j}}_{a_2} e^{j2\pi 880 t} - \underbrace{\frac{1}{2j}}_{a_{-2}} e^{-j2\pi 880 t} \end{cases}$$

* share $\boxed{T = \frac{1}{440}}$

$x(t)$ can be represented by sequence of a_k's:

$k=-3$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	
0	$-\frac{1}{2j}$	$-\frac{1}{2j}$	0	$\frac{1}{2j}$	$\frac{1}{2j}$	0	0	...

$$\downarrow$$

$$\boxed{LTI} \; H(\omega)$$

$$\downarrow$$

new sequence of a_k's.

$$\frac{-H(2\pi 880)}{2j} \quad \frac{-H(2\pi 440)}{2j} \quad 0 \quad \frac{1}{2j} H(2\pi 440) \quad \frac{1}{2j} H(2\pi 880) \quad 0 \quad 0$$

Output is

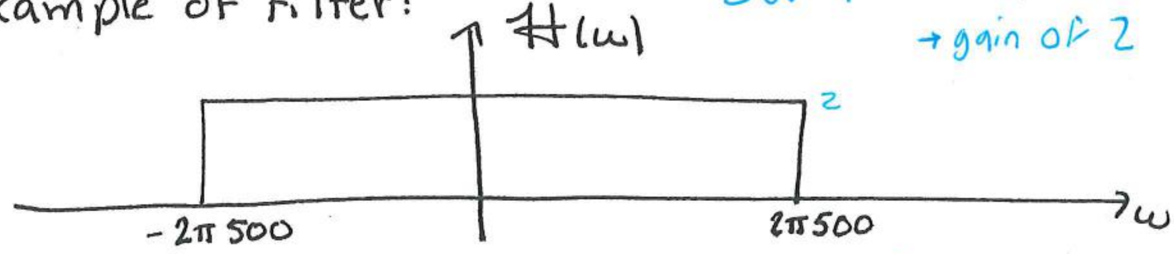
$$y(t) = -\frac{1}{2j} H(-2\pi 880) e^{-j2\pi 880 t} - \frac{1}{2j} H(-2\pi 440) e^{-j2\pi 440 t} + \frac{1}{2j} H(2\pi 440) e^{j2\pi 440 t} + \frac{1}{2j} H(2\pi 880) e^{j2\pi 880 t}$$

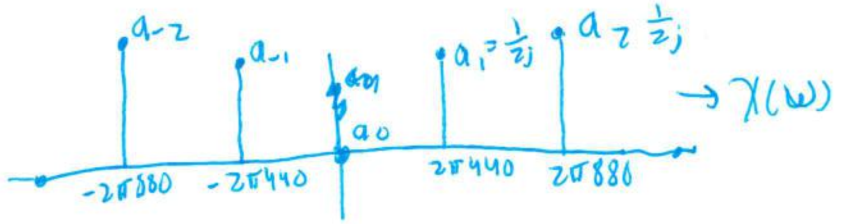
$$a_k \longrightarrow \boxed{LTI} \longrightarrow \tilde{a}_k = a_k H(k\omega_0)$$

Example of filter:

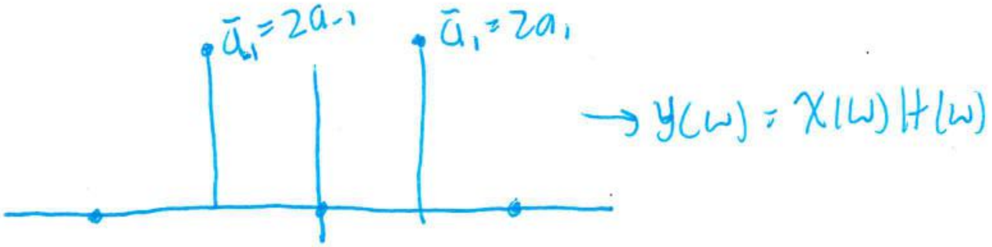
Low Pass Filter

→ gain of 2





* outside of band, dissapear when multiplied
- inside gets multiplied by the gain



* if input signal periodic w/ period $\frac{1}{440}$

→ output signal also period $\frac{1}{440}$, but w/ fewer harmonics

new a_k = old $a_k \cdot H(k\frac{2\pi}{T})$

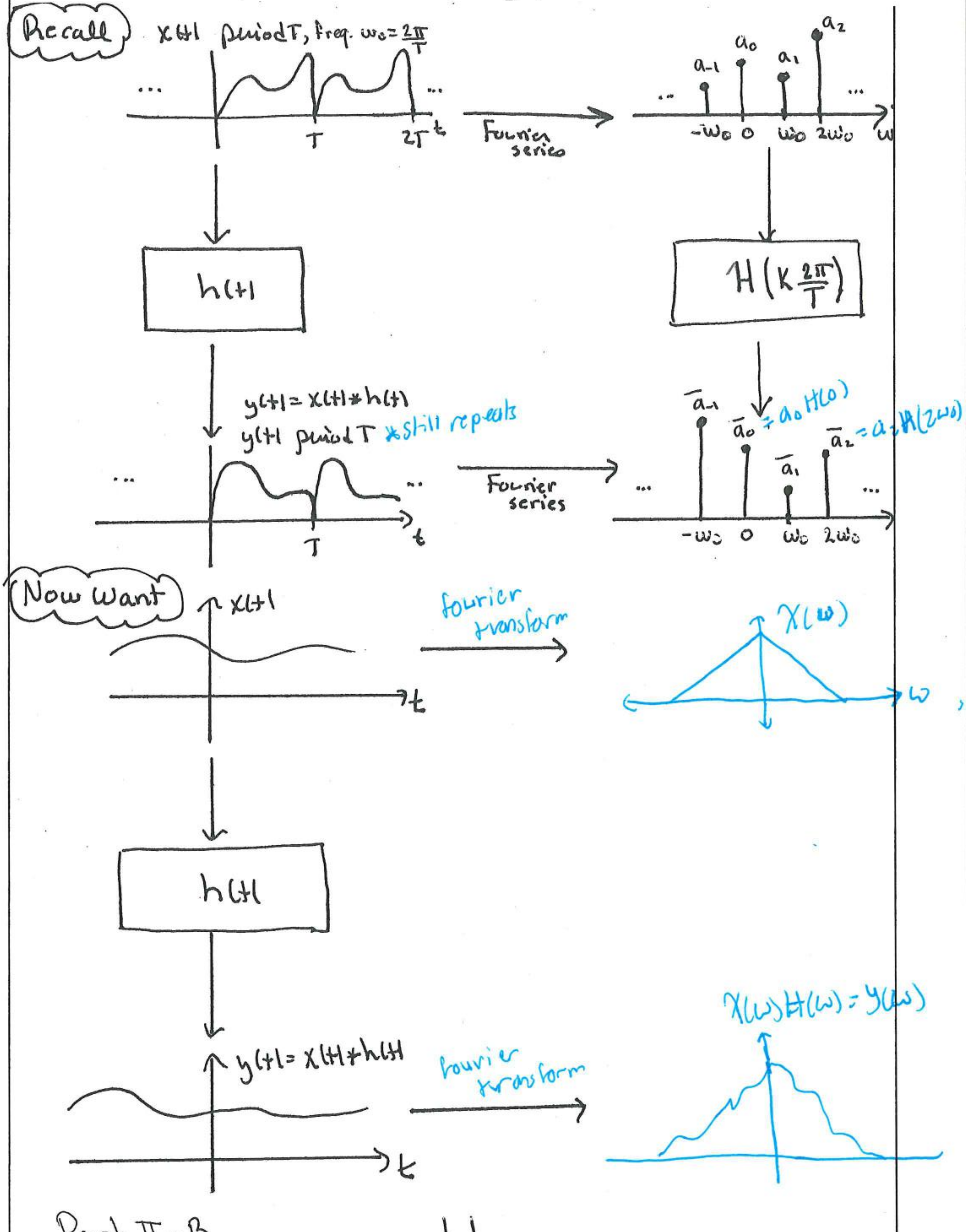
here $H(\omega) = \begin{cases} 1, & |\omega| < 2\pi 500 \\ 0, & \text{else} \end{cases}$

$x(t) =$ linear combo of $\underline{\underline{e^{j\omega t}}}$, $\omega \in \mathbb{R}$

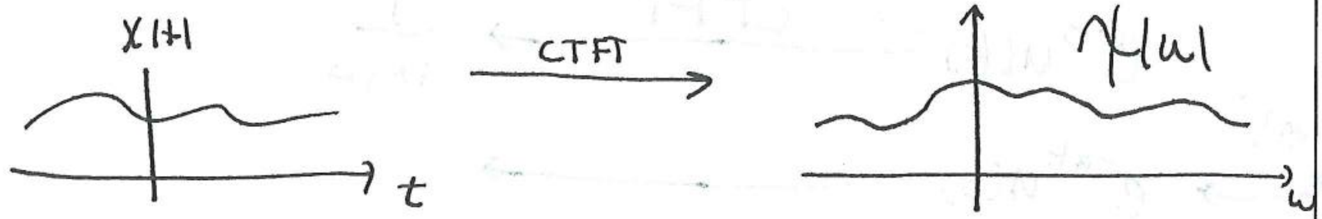
$$\begin{array}{c} \downarrow \\ \boxed{} \\ \downarrow \\ H(\omega)e^{j\omega t} \end{array}$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

1. Why Fourier Transforms?



2. CT Fourier Transform (CTFT): Definition and inverse



$\mathcal{X}(\omega)$ is the projection of the signal $x(t)$ onto the signal $e^{j\omega t}$

Formula CTFT of $x(t)$

$$\boxed{\mathcal{X}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt} = \mathcal{F}(x(t))$$

Intuition for formula:

recall $\mathcal{H}(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$

* freq. response (of LTI) is the fourier transform of $h(t)$

$$e^{j\omega t} \rightarrow \boxed{} \rightarrow \mathcal{H}(\omega)\, e^{j\omega t}$$

obs $\rightarrow$ $\mathcal{H}(\omega)$ = CTFT of $h(t)$

$$e^{-t}u(t) \xrightarrow{\text{CTFT}} \frac{1}{1+j\omega}$$

try!

$$\bigstar \longrightarrow e^{-at}u(t) \longrightarrow$$

$a>0$

$$\delta(t) \xrightarrow{\text{CTFT}} 1$$

$$\delta(t-t_0) \longrightarrow$$

Example1: Compute the CTFT $X(\omega)$ of the signal

$$x(t) = e^{-t}u(t)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-t} u(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-t} e^{-j\omega t}\, dt \quad = \int_{0}^{\infty} e^{-t(1+j\omega)}\, dt$$

$$= \left.\frac{e^{-t(1+j\omega)}}{-(1+j\omega)}\right|_{0}^{\infty}$$

$$= 0 - \frac{-1}{1+j\omega}$$

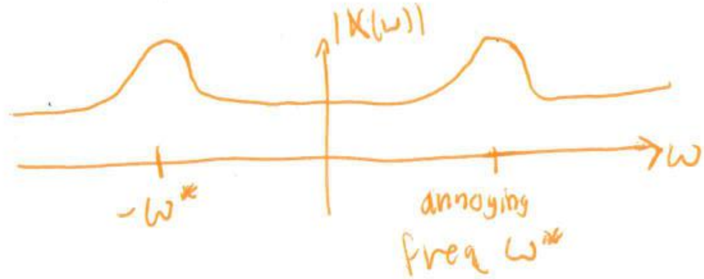
$$= \frac{1}{1+j\omega}$$

a_k is how much of freq $k\omega_0$ there is in the signal $x(t)/x[n]$

how much of " $e^{jk\omega_0 t}/e^{jk\omega_0 n}$ "

$X(\omega)$ is how much frequency ω there is in the signal $x(t)$

how much of $e^{j\omega t}$



Example 2: Compute the CTFT $X(\omega)$ of the signal

$$x(t) = \delta(t).$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega(0)}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t) \cdot 1\, dt$$

$$= 1 \int_{-\infty}^{\infty} \delta(t)\, dt$$

or "by sifting property"

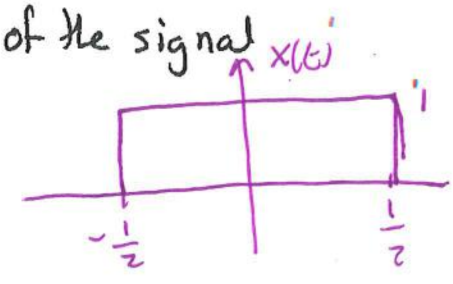
$$= 1$$

$\delta(t-7)$

$e^{-j\omega(7)}$

Example 3: Compute the CTFT $\mathcal{X}(\omega)$ of the signal

$$x(t) = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \text{else} \end{cases}$$



"rect"

$$\mathcal{X}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t}\, dt$$

$$= \begin{cases} \left.\dfrac{e^{-j\omega t}}{-j\omega}\right|_{-\frac{1}{2}}^{\frac{1}{2}}, & \omega \neq 0 \\ 1, & \omega = 0 \end{cases}$$

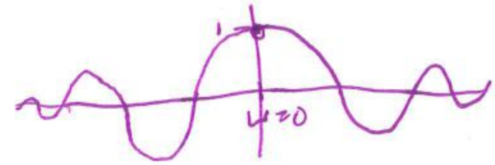
can't ÷ by 0

$$= \begin{cases} \dfrac{e^{-j\frac{\omega}{2}} - e^{j\frac{\omega}{2}}}{-j\omega}, & \omega \neq 0 \\ 1, & \omega = 0 \end{cases}$$

Euler's

$$= \begin{cases} \dfrac{2\sin\left(\frac{\omega}{2}\right)}{\omega}, & \omega \neq 0 \\ 1, & \omega = 0 \end{cases}$$

"sinc"



l'Hopital's rule

$$\lim_{\omega \to 0} \frac{2\sin\left(\frac{\omega}{2}\right)}{\omega} = \lim_{\omega \to 0} \frac{\cos\left(\frac{\omega}{2}\right)}{1} = 1$$

"sinc"

FT of a "rect" is a "sinc"

How to recover $x(t)$ from $X(\omega)$

$$\boxed{\text{Inverse CTFT formula} \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega} = \mathcal{F}^{-1}(X(\omega))$$

weight of component w/ freq ω $e^{j\omega t}$

Intuition behind formula:

$\frac{X(\omega)}{2\pi} \sim$ a_k in fourier series

weight of component $e^{j\omega t}$ in the signal

$\left|\frac{X(\omega^*)}{2\pi}\right|$ Large!

$CTFT\left(\frac{1}{2\pi}\right) \Rightarrow \delta(\omega)$

$$\frac{1}{2\pi} \underset{\text{I CTFT}}{\overset{\text{CTFT}}{\rightleftarrows}} \delta(\omega)$$

Example 1: Compute the inverse CTFT of $X(\omega) = \delta(\omega)$.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi} e^{j\omega t} \Big|_{\omega=0}$$, by sifting property of $\delta(\omega)$

$$= \frac{1}{2\pi}$$

T.S.

$$x(t-t_0) \xrightarrow{CTFT} e^{-j\omega t_0} X(\omega)$$

$$|e^{-j\omega t_0} X(\omega)| = |X(\omega)|$$

same frequency no matter when you "start recording"

→ doesn't change magnitude

$$x^*(t) \xrightarrow{CTFT} X^*(-\omega)$$

general rule

$(F(\omega))^*$ → just replace all j's w/ −j's

* Know proofs for EXAM 2 $\mathcal{F}$ = fourier TF

3. Properties of CTFT

Linearity $\mathcal{F}(ax(t)+by(t)) = a\,\mathcal{F}(x(t)) + b\,\mathcal{F}(y(t))$

for any $a, b \in \mathbb{C}$

*Proof:

$$\int_{-\infty}^{\infty}(ax(t)+by(t))e^{-j\omega t}\,dt = a\int x(t)e^{-j\omega t}\,dt + b\int y(t)e^{-j\omega t}\,dt \quad \square.$$

Time-shifting $\mathcal{F}(x(\underbrace{t-t_0}^{\text{blah}})) = e^{-j\omega t_0}\,\mathcal{F}(x(t))$

for any $t_0 \in \mathbb{R}$ let $\tau = t - t_0$ $d\tau = dt$

*Proof:

$$\int_{-\infty}^{\infty} x(\underbrace{t-t_0}_{\text{blah}})e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_0)}\,d\tau$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,d\tau$$

$$= e^{-j\omega t_0}\,\mathcal{F}(x(t)) \quad \square.$$

Conjugation and conjugate symmetry

If $\mathcal{F}(x(t)) = X(\omega)$, then $\mathcal{F}(x(t)^*) = X^*(-\omega)$.

*Proof:

$$X^*(-\omega) = \left(\int_{-\infty}^{\infty} x(t)e^{+j\omega t}\,dt\right)^* \leftarrow \text{conjugate everything inside}$$

$$= \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}\,dt$$

$$= \mathcal{F}(x(t)^*) \quad \square$$

This property implies that
- The real part of the F.T. of a real signal is even.
- The imaginary " " " " " " odd.

Also
- The F.T. of a real and even signal is real and even.

-chose $x(-t)$ so that $x(t) = x(-t)$ *make it even! → reflect over y-axis

$\underline{x(t) \text{ real + even then } X(\omega) \text{ is also real + even}}$

Part II-B

why? because if $x(t)$ is real

then $x^*(t) = x(t)$

$$\Rightarrow \quad \mathcal{F}(x^*(t)) = \mathcal{F}(x(t))$$

$$\Rightarrow \quad \mathcal{X}^*(-\omega) = \mathcal{X}(\omega) \text{, by conj. property}$$

$$\Rightarrow \quad Re(\mathcal{X}(-\omega)) - j\, Im(\mathcal{X}(-\omega)) = Re(\mathcal{X}(\omega)) + j\, Im(\mathcal{X}(\omega))$$

$$\Rightarrow \quad Re\, \mathcal{X}(-\omega) = Re(\mathcal{X}(\omega)) \quad \text{even}$$

$$-Im(\mathcal{X}(-\omega)) = Im(\mathcal{X}(\omega)) \quad \text{odd} \quad \square$$

If, in addition, $x(t)$ is even

$$\Rightarrow x(-t) = x(t).$$

and therefore

$$\mathcal{X}(-\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(-\tau)\, e^{-j\omega \tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega \tau}\, d\tau$$

$$= \mathcal{X}(\omega)$$

so $\mathcal{X}(\omega)$ is even

but

$$\underbrace{\mathcal{X}(\omega)}_{\text{even}} = \underbrace{Re(\mathcal{X}(\omega))}_{\text{even}} + j\, \underbrace{Im\, \mathcal{X}(\omega)}_{\text{odd}}$$

so odd part must be zero.

$\Rightarrow \mathcal{X}(\omega)$ real. $\square$

★ could be asked to prove on exam!

Differentiation and integration If $\mathcal{F}(x(t)) = X(\omega)$

(1) $\frac{d}{dt} x(t) \xrightarrow{CTFT} j\omega X(\omega)$ ★ used a lot! (later)

(2) $\int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{CTFT} \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$

Time scaling If $\mathcal{F}(x(t)) = X(\omega)$

then $\mathcal{F}(x(at)) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$, $a \neq 0$

$x(2t)$ squeeze

$X\left(\frac{\omega}{2}\right)$ expand

Proof: $\mathcal{F}(x(at)) = \int_{-\infty}^{\infty} x(at) e^{-j\omega t}\, dt$ let $\tau = at$

$$= \begin{cases} \int_{-\infty}^{\infty} x(\tau) e^{\frac{-j\omega\tau}{a}} \frac{d\tau}{a}, & a > 0 \\ \int_{\infty}^{-\infty} x(\tau) e^{-\frac{j\omega\tau}{a}} \frac{d\tau}{a}, & a < 0 \end{cases} = \begin{cases} \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\frac{\tau}{a}}\, d\tau, & a > 0 \\ -\frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\frac{\tau}{a}}\, d\tau, & a < 0 \end{cases}$$

$\square$

Frequency Scaling

$$X(b\omega) \xrightarrow{I\,CTFT} \frac{1}{|b|} x\left(\frac{t}{b}\right), \quad b \neq 0$$

proof let $b = \frac{1}{a}$ in scaling property

$x(at) \xrightarrow{CTFT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

"therefore" $\Rightarrow x\left(\frac{t}{b}\right) \xrightarrow{CTFT} |b| X(b\omega)$

by linearity, multiply both sides by $\frac{1}{|b|}$

$\Rightarrow |b|\, x\left(\frac{t}{b}\right) \xrightarrow{CTFT} X(b\omega)$

$\square$

Duality If $x(t) \xrightarrow{\text{CTFT}} X(\omega)$

then $X(t) \xrightarrow{\text{CTFT}} 2\pi\, x(-\omega)$

Proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega, \quad \text{let } \tau = \omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)\, e^{j\tau t}\, d\tau$$

$$\Rightarrow x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau)\, e^{j\tau\omega}\, d\tau$$

$$\Rightarrow 2\pi\, x(-\omega) = \frac{2\pi}{2\pi}\int_{-\infty}^{\infty} X(\tau)\, e^{-j\tau\omega}\, d\tau$$

let $t = \tau$

$$= \int_{-\infty}^{\infty} \underbrace{X(t)}_{\text{"blah"}}\, e^{-jt\omega}\, dt = \text{CTFT}(X(t)) \quad \square$$

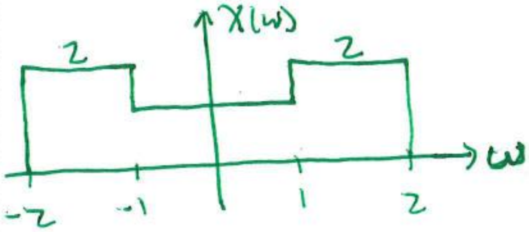
Parseval's Relation

E_∞ total energy of $x(t)$ — energy of CTFT $X(\omega)$

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Ex Q.1

A signal $x(t)$ has CTFT given by following graph



Does $x(t)$ have finite energy?

Ans) $E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$, by Parseval's Relation

$$= \frac{1}{2\pi}\left(\int_{-2}^{-1} 2^2 d\omega + \int_{-1}^{1} 1^2 d\omega + \int_{1}^{2} 2^2 d\omega\right)$$

$$= \frac{1}{2\pi}(4+2+4)$$

$$= \frac{5}{\pi}$$

so E_∞ is finite

* can use if asked to evaluate difficult integral

$$\int_{-\infty}^{\infty} \sin^2 t \, dt = \int_{-\infty}^{\infty} |CTFT(\sin t)|^2 dt$$

two direct-deltas

recall an LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow x(t) * h(t) = y(t)$$

$$x(t) = \frac{1}{2\pi} \int X(\omega)\, e^{j\omega t}\, d\omega$$

$$y(t) = \frac{1}{2\pi} \int Y(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int X(\omega) H(\omega)\, e^{j\omega t}\, d\omega$$

Convolution Property

$$x_1(t) * x_2(t) \xrightarrow{\text{CTFT}} X_1(\omega) X_2(\omega)$$

So for LTI systems, the CTFT of the output $y(t)$ is

$$Y(\omega) = X(\omega) H(\omega)$$

frequency response.

and

$$y(t) = \mathcal{F}^{-1}\left(X(\omega) H(\omega) \right)$$

Recall Properties of LTI systems

<u>in time domain</u> | <u>in frequency domain</u>

(1) $x(t) \rightarrow \boxed{h(t)} \rightarrow x(t) * h(t)$ $\qquad$ $X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow X(\omega) H(\omega) = Y(\omega)$

(2) $x(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow y(t)$ $\qquad$ $X(\omega) \rightarrow \boxed{H_1(\omega)} \rightarrow \boxed{H_2(\omega)} \rightarrow Y(\omega)$

same as

$x(t) \rightarrow \boxed{h_1(t) * h_2(t)} \rightarrow y(t)$ $\qquad$ $X(\omega) \rightarrow \boxed{H_1(\omega) H_2(\omega)} \rightarrow Y(\omega).$

multiplication property

$$x_1(t)\, x_2(t) \xrightarrow{\text{CTFT}} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

Example:

$$x(t)\, e^{j\omega_0 t} \xrightarrow{\text{CTFT}} \frac{1}{2\pi} X(\omega) * 2\pi\, \delta(\omega-\omega_0) = X(\omega) * \delta(\omega-\omega_0)$$

$$= X(\omega-\omega_0)$$

phase shift

freq. shift

4. Fourier transform of CT periodic signals

If signal periodic with period T $\qquad T = \frac{2\pi}{\omega_0}$

$$\Rightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \text{where } \omega_0 = \frac{2\pi}{T}$$

What is the Fourier transform of this fourier series?

$$\mathcal{F}(x(t)) = \mathcal{F}\left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right)$$

$$= \sum_{k=-\infty}^{\infty} a_k \, \mathcal{F}(e^{jk\omega_0 t}), \quad \text{by linearity of CTFT}$$

So all we need to know is $\mathcal{F}(e^{j\omega_0 t})$, $\omega_0 \in \mathbb{R}$.

but,

$$\cancel{\mathcal{F}(e^{jk\omega_0 t})} = \int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{\infty} e^{jt(k\omega_0 - \omega)} \, dt$$

* don't try to CTFT a periodic signal!

$$= \begin{cases} \dfrac{e^{jt(k\omega_0 - \omega)}}{j(k\omega_0 - \omega)}, & \omega_0 \neq \omega \quad ? \\ \int_{\infty}^{\infty} 1 \, dt, & \omega_0 = \omega \quad \infty \end{cases}$$

- function must "taper down" to 0 in order to integrate

CTFT of $x(t) = 1$

$$\int_{-\infty}^{\infty} 1 e^{-j\omega t} \, dt$$

guess $\mathcal{F}\left(e^{j\omega_0 t}\right) = 2\pi\,\delta(\omega-\omega_0)$

check: $\mathcal{F}^{-1}\left(2\pi\,\delta(\omega-\omega_0)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0)\,e^{j\omega t}\,d\omega$

$$= \int_{-\infty}^{\infty} e^{j\omega_0 t}\,\delta(\omega-\omega_0)\,d\omega$$

$$= e^{j\omega_0 t}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\,d\omega$$

$$= e^{j\omega_0 t}$$

so

$$\boxed{e^{j\omega_0 t} \xrightarrow{\text{CTFT}} 2\pi\,\delta(\omega-\omega_0)}$$

$$x(t) = e^{j\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(\omega)}_{2\pi\delta(\omega-\omega_0)}\, e^{j\omega t}\,d\omega$$

$$\underbrace{\sin\omega_0 t}_{\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{2j}} \xrightarrow{\text{CTFT}} \frac{2\pi\,\delta(\omega-\omega_0) - 2\pi\,\delta(\omega-\omega_0)}{2j} = \frac{\pi}{j}\delta(\omega-\omega_0) - \frac{\pi}{j}\delta(\omega+\omega_0)$$

$$\cos\omega_0 t \longrightarrow \pi\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)$$

5. Frequency Response of CT LTI systems

Recall: for LTI systems

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow e^{j\omega t} H(\omega)$$

$$\text{where } H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$$

$H(\omega)$ is the CTFT of $h(t)$

Now we know that, more generally

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) H(\omega) e^{j\omega t}\, d\omega$$

or, viewing system in frequency domain

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega) = X(\omega) H(\omega)$$

Example: $x(t) = 1$

$h(t) = \delta(t - t_0)$

$$Y(\omega) = \mathcal{F}(x(t))\,\mathcal{F}(h(t))$$

$$= \mathcal{F}(1)\,\mathcal{F}(\delta(t-t_0))$$

$$= 2\pi\,\delta(\omega) \cdot e^{-j\omega t_0}\,\mathcal{F}(\delta(t))$$

$$= 2\pi\,\delta(\omega)\,e^{-j\omega t_0} \cdot 1$$

$$= 2\pi\,\underline{e^{j\omega_0 t_0}}\,\delta(\omega)$$

$$= 2\pi\,\delta(\omega)$$

$$\Big\downarrow \text{ICTFT}$$

$$1$$

Same as:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau = \int_{-\infty}^{\infty} \delta(\tau - t_0)\,dt = 1$$

Probs on test!

One can define an LTI system using constant coefficient differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} x(t)$$

Examples: $\mathcal{H}(\omega) = ?$

We will use $Y(\omega) = X(\omega)\mathcal{H}(\omega)$

$$+ \quad \frac{d}{dt} x(t) \xrightarrow{CTFT} j\omega X(\omega)$$

$$\frac{d}{dt}\left(\frac{d}{dt} x(t)\right) \xrightarrow{CTFT} j\omega\left(j\omega X(\omega)\right)$$

EX

$$12 \frac{d}{dt} y(t) + 7 y(t) = x(t)$$

(CTFT of both sides)

$$12 j\omega Y(\omega) + 7 Y(\omega) = X(\omega)$$

$$Y(\omega)(12 j\omega + 7) = X(\omega)$$

$$Y(\omega) = X(\omega) \frac{1}{12 j\omega + 7} \Rightarrow H(\omega) = \frac{1}{12 j\omega + 7}$$

Trick to obtain frequency response of CT LTI systems defined by a differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} x(t)$$

$$\Rightarrow \mathcal{F}\left(\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t)\right) = \mathcal{F}\left(\sum_{m=0}^{M} b_m \frac{d^m}{dt^m} x(t)\right)$$

$$\Rightarrow \sum_{k=0}^{N} a_k \mathcal{F}\left(\frac{d^k}{dt^k} y(t)\right) = \sum_{m=0}^{M} b \mathcal{F}\left(\frac{d^m}{dt^m} x(t)\right)$$

$$\Rightarrow \sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{m=0}^{M} b_m (j\omega)^m X(\omega)$$

$$\Rightarrow \qquad Y(\omega) = \underbrace{\frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{k=0}^{N} a_k (j\omega)^k}}_{H(\omega)} X(\omega)$$

So

$$\boxed{H(\omega) = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{k=0}^{N} a_k (j\omega)^k}}$$

1. DT Fourier transform: definition and inverse

$$x[n] \xrightarrow{\text{DTFT}} X(\omega)$$

$$X(\omega) \xrightarrow{\text{inverse DTFT}} x[n]$$

Formulas

$$\boxed{\begin{array}{ll} \text{DTFT} & X(\omega) = \mathcal{F}(x[n]) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \omega \in \mathbb{R} \\ \text{IDTFT} & x[n] = \mathcal{F}^{-1}(X(\omega)) = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{X(\omega) e^{j\omega n}}_{\text{periodic w/ } 2\pi} d\omega \end{array}}$$

has to have length 2π

Example:

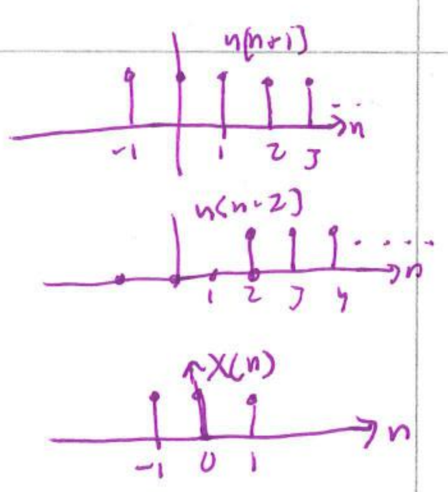
$x[n] = 3^{-n} u[n]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} 3^{-n} u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} 3^{-n} e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{3e^{j\omega n}}\right)^n$$

since $\left|\frac{1}{3e^{j\omega}}\right| = \frac{1}{3} < 1$

$$\Rightarrow X(\omega) = \frac{1}{1 - \frac{1}{3e^{j\omega}}} = \frac{3e^{j\omega}}{3e^{j\omega} - 1}$$

example: $x[n] = u[n+1] - u[n-2]$



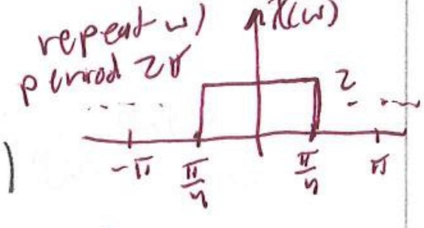
$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-1}^{1} e^{-j\omega n}$$

$$= e^{j\omega} + e^{0} + e^{-j\omega}$$

$$= 1 + 2\cos\omega$$

2. Properties of the DTFT



periodicity

$$X(\omega+2\pi) = X(\omega)$$

Proof: $X(j\omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi)n}$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \underbrace{e^{-j2\pi n}}_{=1}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = X(\omega) \checkmark$$

Linearity

$$a x_1[n] + b x_2[n] \xrightarrow{\text{DTFT}} a X_1(\omega) + b X_2(\omega)$$

Proof: $\mathcal{F}(a x_1[n] + b x_2[n]) = \sum_{n=-\infty}^{\infty} (a x_1[n] + b x_2[n])\, e^{j\omega n}$

$$= a \sum_{n=-\infty}^{\infty} x_1[n]\, e^{-j\omega n} + b \sum_{n=-\infty}^{\infty} x_2[n]\, e^{-j\omega n}$$

$$= a\, \mathcal{F}(x_1[n]) + b\, \mathcal{F}(x_2[n]).$$

$\square$

$$e^{j\omega_0 n} x[n] \longrightarrow X(\omega-\omega_0)$$

$$\mathcal{F}\left(e^{j\omega_0 n} x[n]\right) = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega-\omega_0)n}$$

$$= X(\omega-\omega_0)$$

✓

time shifting and frequency shifting

$$x[n-n_0] \xrightarrow{\text{DTFT}} e^{-j\omega n_0} X(\omega)$$

$$e^{j\omega_0 n}\, x[n] \xrightarrow{\text{DTFT}} X(\omega-\omega_0)$$

Proof: $\mathcal{F}(x[n-n_0]) = \sum_{n=-\infty}^{\infty} x[\underbrace{n-n_0}_{=m}]\, e^{-j\omega n}$

let $m = n - n_0$

$$= \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\omega(m+n_0)}$$

$$= \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\omega m}\, \underline{e^{-j\omega n_0}}$$

$$= e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\omega m}$$

$$= e^{-j\omega n_0} X(\omega)$$

note:

$$|e^{-j\omega n_0} X(\omega)| = |X(\omega)|$$

hw

$X(\omega) = \sin\omega$

$= \underbrace{\sin(\omega)}_{\text{odd}} + j\underbrace{0}_{\substack{\text{even}\\ \text{odd}}}$

so signal $x[n]$ not real

*helps w/ hw 7

Conjugation and conjugate symmetry

$$x^*[n] \xrightarrow{\text{DTFT}} X^*(-\omega)$$

Proof: if $x[n]$ real

i.e. $x^*[n] = x[n]$

$\Rightarrow X^*(-\omega) = X(\omega)$

$\underbrace{Re(-\omega)}\overbrace{-jIm(-\omega)} = \underbrace{Re(\omega)}\overbrace{+jIm(\omega)}$ multiply both sides by j

$x(n)$ real

$Re(-\omega) = Re(\omega) \leftarrow$ $Re(\omega)$ even

$-Im(-\omega) = Im(\omega) \leftarrow$ $Im(\omega)$ odd

$y(n)$ pure imag.

$Re(\omega)$ odd

$Im(\omega)$ even

$jRe(-\omega) = jRe(\omega)$

$-Im(-\omega) = Im(\omega)$

Important Corollary: if $x[n]$ is real, then $X(\omega) = X^*(-\omega)$.

$\Rightarrow Re(X(\omega))$ even function of ω

$Im(X(\omega))$ odd function of ω

Parseval's Relation

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{\text{energy of signal}} = \frac{1}{2\pi} \underbrace{\int_{0}^{2\pi} |X(\omega)|^2 \, d\omega}_{\text{energy of F.T.}}$$

Convolution Property

$$x_1[n] * x_2[n] \xrightarrow{\text{DTFT}} X_1(\omega)\, X_2(\omega)$$

$$x[n] \longrightarrow \overset{\text{LTI}}{\boxed{h(n)}} \longrightarrow y(n) = x[n] * h[n]$$

$$\Big\downarrow \text{DTFT}$$

$$X(\omega) \qquad\qquad Y(\omega) = X(\omega)\, H(\omega)$$

Multiplication Property

$$x[n]\,y[n] \xrightarrow{\text{DTFT}} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

No duality in DT!

Differentiation in frequency

$$n\,x[n] \xrightarrow{\text{DTFT}} j \frac{d}{d\omega} X(\omega).$$

3. Fourier transform of DT periodic signals

$x[n]$ periodic period N $\Rightarrow$ $x[n] = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}$

$$\Rightarrow \mathcal{F}\left(x[n]\right) = \mathcal{F}\left(\sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}\right)$$

$$= \sum_{k=0}^{N-1} a_k \, \mathcal{F}\left(e^{jk\frac{2\pi}{N}n}\right).$$

So all we need to know is

$$\mathcal{F}\left(e^{jk\frac{2\pi}{N}n}\right) = ? \quad , \quad k=0,1,2,\ldots,N-1$$

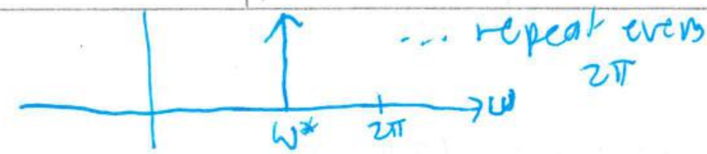
Let's try with the definition of DTFT: Bad! do NOT do!

$$\cancel{= \sum_{n=-\infty}^{\infty} e^{jk\frac{2\pi}{N}n} e^{-j\omega n}}$$

$$\cancel{= \sum_{n=-\infty}^{\infty} e^{j(k\frac{2\pi}{N}-\omega)n} = \sum_{n=-\infty}^{\infty} \left(e^{j(k\frac{2\pi}{N}-\omega)n}\right)}$$

diverges

Let $\omega^* = k\frac{2\pi}{N}$,



observe $\omega^* \in [0, 2\pi)$ since $k = 0, 1, 2, \ldots, N-1$

guess
$2\pi = C$

$$X(\omega) = \begin{cases} 2\pi\,\delta(\omega-\omega^*), & \text{for } \omega \in [0, 2\pi) \\ \text{repeat periodically with period } \underline{\underline{2\pi}}, & \text{for other } \omega. \end{cases}$$

Check

$$\mathcal{F}^{-1}(X(\omega)) = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{\cancel{2\pi}}\int_0^{2\pi} \cancel{2\pi}\,\delta(\omega-\omega^*)\, e^{j\omega n}\, d\omega$$

$$= e^{j\omega^* n} \quad \text{(by sifting property of } \delta(t))$$

$$=$$

so

$$e^{j\omega^* n} \xrightarrow{\text{CTFT}} \ldots + 2\pi\delta(\omega-\omega^*-2\pi) + 2\pi\delta(\omega-\omega^*) + 2\pi\delta(\omega-\omega^*+2\pi) + 2\pi\delta(\omega-\omega^*+4\pi) + \ldots$$

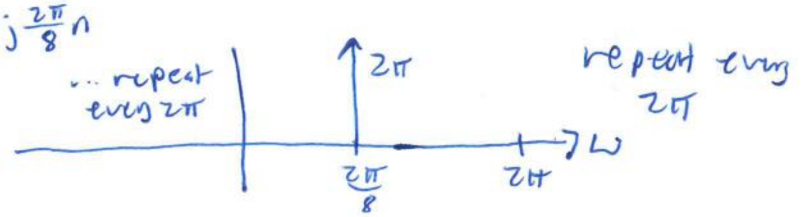
$$= \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega^*+2\pi k)$$

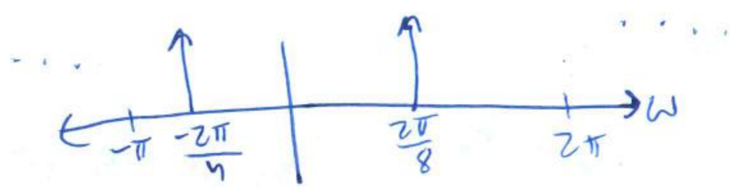
$$x[n] = \sin\left(\frac{2\pi}{8} n\right)$$

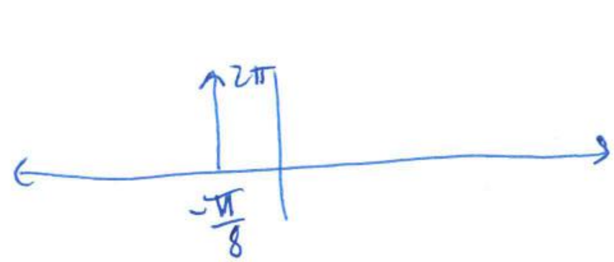
$N = 8$

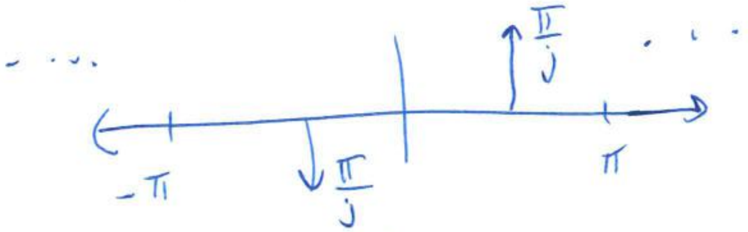
$$= \frac{e^{j\frac{2\pi}{8}n} - e^{-j\frac{2\pi}{8}n}}{2j} = \frac{1}{2j} e^{j\frac{2\pi}{8}n} - \frac{1}{2j} e^{-j\frac{2\pi}{8}n}$$

DTFT of $e^{j\frac{2\pi}{8}n}$









$$= 2\pi \cdot \frac{1}{2j} = \frac{\pi}{j}$$

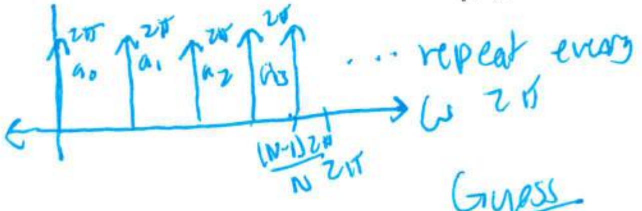
$$\sin\left(\frac{2\pi}{8} n\right) \xrightarrow{\text{DTFT}} \frac{\pi}{j}\delta\left(\omega - \frac{2\pi}{8}\right) - \frac{\pi}{j}\delta\left(\omega + \frac{2\pi}{8}\right)$$

repeat every 2π

$$= \sum_{l=-\infty}^{\infty}\left(\frac{\pi}{j}\delta\left(\omega - \frac{2\pi}{8} + 2\pi l\right) - \frac{\pi}{j}\delta\left(\omega + \frac{2\pi}{8} + 2\pi l\right)\right)$$

★ MUST repeat every 2π

Therefore $\mathcal{F}\left(\sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}\right) = \sum_{k=0}^{N-1} a_k \mathcal{F}\left(e^{jk\frac{2\pi}{N}n}\right)$



$$= \sum_{k=0}^{N-1} a_k \sum_{l=-\infty}^{\infty} 2\pi\,\delta\left(\omega - k\tfrac{2\pi}{N} + 2\pi l\right)$$

Guess

$$X(\omega) = \sum_{k=0}^{N-1} a_k 2\pi\,\delta\left(\omega - k\tfrac{2\pi}{N}\right), \text{ for } \omega \in [0, 2\pi)$$

repeat w/ period 2π for other ωs

$$= \sum_{k=-\infty}^{\infty} a_k 2\pi\,\delta\left(\omega - k\tfrac{2\pi}{N}\right)$$

★ $a_{k+N} = a_k$

Obs

$$\mathcal{F}^{-1}\left(\sum_{k=-\infty}^{\infty} a_k 2\pi\,\delta\left(\omega - k\tfrac{2\pi}{N}\right)\right) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^{N-1} a_k 2\pi\,\delta\left(\omega - k\tfrac{2\pi}{N}\right) e^{j\omega n}\, d\omega$$

$$= \sum_{k=0}^{N-1} a_k \underbrace{\int_0^{2\pi} \delta\left(\omega - k\tfrac{2\pi}{N}\right) e^{j\omega n}\, d\omega}_{= e^{j\frac{k\pi}{N}n}\text{, by sifting prop.}}$$

$$= \underbrace{\sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}}_{x(n)}$$

$$\boxed{\sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n} \xrightarrow{\text{DTFT}} \sum_{k=-\infty}^{\infty} 2\pi a_k\,\delta\left(\omega - k\tfrac{2\pi}{N}\right)}$$

4. Frequency Response of DT LTI systems

Recall: for LTI systems

$$e^{j\omega n} \rightarrow \boxed{h[n]} \rightarrow e^{j\omega n} H(\omega)$$

where $H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$

$H(\omega)$ is the DTFT of $h[n]$

Now we know that, more generally

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{j\omega t} d\omega \rightarrow \boxed{h[n]} \rightarrow y[n] = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) H(\omega) e^{j\omega t} d\omega$$

or, viewing system in frequency domain

$$X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow Y(\omega) = X(\omega) H(\omega)$$

CT $e^{j\omega t} \rightarrow \boxed{} \rightarrow e^{j\omega t} H(\omega)$

$e^{st} \rightarrow \boxed{} \rightarrow e^{st} H(s)$

$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$ freq resp

$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$ T.F.

$H(j\omega) = H(\omega)$

$X(\omega) = \underline{\underline{X}}(e^{j\omega})$

$X(s)$ Laplace Trans.

$H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$ freq. resp.

$e^{j\omega n} \rightarrow \boxed{} \rightarrow e^{j\omega n} H(\omega)$ freq resp.

$z^n \rightarrow \boxed{} \rightarrow z^n H(z)$ transfer func.

DT $H(\omega) = H(e^{j\omega})$

One can define a causal LTI system using a constant coefficient difference equation

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{m=0}^{M} b_m \, x[n-m]$$

Examples: $y[n] - \frac{3}{4} y[n-1] + \frac{1}{8} y[n-2] = 2x[n]$

$\mathcal{F}($ L.H.S. $) = \mathcal{F}($R.H.S.$)$

$$Y(\omega) - \frac{3}{4}\mathcal{F}(y[n-1]) + \frac{1}{8}\mathcal{F}(y[n-2]) = 2X(\omega)$$

recall $\mathcal{F}(x[n-n_0]) = e^{-j\omega n_0} X(\omega)$

$$\Rightarrow Y(\omega) - \frac{3}{4} e^{-j\omega} Y(\omega) + \frac{1}{8} e^{-2j\omega} Y(\omega) = 2X(\omega)$$

$$Y(\omega)\left(1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{-2j\omega}\right) = 2X(\omega)$$

$$Y(\omega) = \underbrace{\frac{2}{1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{-2j\omega}}}_{H(\omega)} X(\omega)$$

Part II-C

(-) sgn

$$\mathcal{F}(x[n-1]) = \sum_{n=-\infty}^{\infty} x[n-1] e^{-j\omega n}$$

let $n = m-1$

$$\sum_{m=-\infty}^{\infty} x[m] e^{-j\omega(m+1)}$$

$$= e^{-j\omega n} X(\omega)$$

* TEST !!! | P.F. "hardest part"

-skeleton of solution

Trick to obtain the frequency response of DT LTI systems defined by difference equations.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

$$\Rightarrow \mathcal{F}\left(\sum_{k=0}^{N} a_k y[n-k] \right) = \mathcal{F}\left(\sum_{m=0}^{M} b_m x[n-m] \right)$$

$$\Rightarrow \sum_{k=0}^{N} a_k \mathcal{F}\left(y[n-k] \right) = \sum_{m=0}^{M} b_m \mathcal{F}\left(x[n-m] \right)$$

$$\Rightarrow \sum_{k=0}^{N} a_k e^{-j\omega k} \, \underline{\underline{Y(\omega)}} = \sum_{m=0}^{M} b_m e^{-j\omega m} \, \underline{\underline{X(\omega)}}$$

$$\Rightarrow Y(\omega) = \underbrace{\frac{\sum_{m=0}^{M} b_m e^{-j\omega m}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}}_{H(\omega)} X(\omega)$$

So $$\boxed{H(\omega) = \frac{\sum_{m=0}^{M} b_m e^{-j\omega n}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}}$$

Find unit impulse response?

write $H(\omega)$ → P.F.D partial fraction

$$H(\omega) = \frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}}$$

$\underline{\text{hint:}}$ $a^n u(n) \xrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}$

$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

<u>scratch paper</u>

$$\frac{2}{1-\frac{3}{4}x+\frac{1}{8}x^2} = \frac{2}{(1-\frac{1}{2}x)(1-\frac{1}{4}x)}$$

$$= \frac{A}{1-\frac{1}{2}x} + \frac{B}{1-\frac{1}{4}x} \quad \text{P.F.D} \rightarrow \begin{matrix} A=4 \\ B=-2 \end{matrix}$$

* if run out of points on test

↳ just leave A + B as variables (only lose few pts)

$$= \frac{A}{1-\frac{1}{2}x} + \frac{B}{1-\frac{1}{4}x}$$

$$h[n] = A\left(\tfrac{1}{2}\right)^n u(n) + B\left(\tfrac{1}{4}\right)^n u[n]$$

$$h[n] = 4\left(\tfrac{1}{2}\right)^n u(n) + (-2)\left(\tfrac{1}{4}\right)^n u[n]$$

$$x[n] = a^n u[n]$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$|a|<1$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \begin{cases} \frac{1}{1-ae^{-j\omega}}, & \text{if } |ae^{-j\omega}|<1 \\ , & \text{else} \end{cases}$$

but $|ae^{-j\omega}| = |a|\,|e^{-j\omega}|$

$= |a| \cdot 1$

$= |a|$