ECE 544 Fall 2013
Problem Set 9
Due November 22, 2013

1. Read Chapter 7 of M. B. Pursley, *Introduction to Digital Communications* (MBP).
2. MBP Problems 7.2, 7.3, 7.4, 7.6

The "left over problems" from Problem Set 8

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7.2 Show that for any real numbers α and β,

$$\int_0^{2\pi} \exp\{\alpha\cos(\varphi) - \beta\sin(\varphi)\}\, \mathrm{d}\varphi/2\pi = I_0\left(\sqrt{\alpha^2+\beta^2}\right).$$

Hint: Write $\alpha\cos(\varphi) - \beta\sin(\varphi)$ as $\sqrt{\alpha^2+\beta^2}\cos(\varphi+\psi)$, where ψ is defined by

$$\sin(\psi) = \beta\big/\sqrt{\alpha^2+\beta^2}$$

and

$$\cos(\psi) = \alpha\big/\sqrt{\alpha^2+\beta^2}.$$

In other words, $\psi = \tan^{-1}(\beta/\alpha)$. This shows that the original integral is equal to

$$\int_0^{2\pi} \exp\{\sqrt{\alpha^2+\beta^2}\cos(\varphi+\psi)\}\, \mathrm{d}\varphi/2\pi.$$

Now use the fact that $\cos(\theta)$ is periodic in θ to obtain the desired result.

MBP 7.2 Solution from Lecture Notes

Relating Integral to $I_0(z)$ modified Bessel function of first Kind of order zero

$$I_0(z) = \int_0^{2\pi} \exp\{z\cos\theta\} \frac{d\theta}{2\pi}$$

Properties: $I_0(-z) = I_0(z)$

$|z_1| < |z_2| \implies I_0(z_1) < I_0(z_2)$

Then in (**) let $a = \alpha_0 u_0/\sigma_0^2$, $b = \alpha_0 v_0/\sigma_0^2$ and define

$$\cos\psi = \frac{a}{\sqrt{a^2+b^2}} \qquad \sin\psi = \frac{b}{\sqrt{a^2+b^2}}$$

Then

$$(**) = \frac{1}{2\pi}\int_0^{2\pi} \exp\{a\cos\varphi - b\sin\varphi\}\, d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \exp\Big\{\sqrt{a^2+b^2}\,\underbrace{\left[\cos\psi\cos\varphi - \sin\psi\sin\varphi\right]}_{\cos(\psi+\varphi)}\Big\}\, d\varphi$$

$$(**) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left\{\sqrt{a^2+b^2}\cos(\psi+\varphi)\right\} d\varphi$$

make c.o.v.

$$\theta = \psi + \varphi$$

and use periodicity of cosine

$$= \frac{1}{2\pi}\int_0^{2\pi} \exp\left\{\sqrt{a^2+b^2}\cos\theta\right\} d\theta$$

$$= I_0\left(\sqrt{a^2+b^2}\right) = \cancel{I_0\left(\alpha_0\sqrt{u_0^2+v_0^2}/\sigma_0^2\right)}$$

$$\cancel{\therefore\ f_0(u_0,v_0,u_1,v_1) = g_{\sigma_1}(u_1)\,g_{\sigma_1}(v_1)\,g_{\sigma_0}(u_0)\,g_{\sigma_0}(v_0)\,e^{-\alpha_0^2/2\sigma_0^2}\,I_0\left(\alpha_0\sqrt{u_0^2+v_0^2}/\sigma_0^2\right).}$$

~~In the same way~~

$$\cancel{f_1(u_0,v_0,u_1,v_1) = g_{\sigma_0}(u_0)\,g_{\sigma_0}(v_0)\,g_{\sigma_1}(u_1)\,g_{\sigma_1}(v_1)\,e^{-\alpha_1^2/2\sigma_1^2}\,I_0\left(\alpha_1\sqrt{u_1^2+v_1^2}/\sigma_1^2\right)}$$

7.3 In Section 7.5, the random variable R_0 is defined by

$$R_0 = \sqrt{U_0^2 + V_0^2},$$

where U_0 and V_0 are independent Gaussian random variables, $\mathrm{Var}\{U_0\} = \mathrm{Var}\{V_0\} = \sigma^2$, $E\{U_0\} = u_0$, and $E\{V_0\} = v_0$.

(a) For the random variable R_0 in place of R_1, repeat the steps used in going from (7.42) to (7.49) and apply the results of Problem 7.2 to conclude that the density function for R_0 is given by

$$f_{R_0}(r) = \begin{cases} (r/\sigma^2)\exp\{-(r^2 + c^2)/2\sigma^2\}\, I_0(cr/\sigma^2), & r \geq 0, \\ 0, & r < 0, \end{cases}$$

where $c = \sqrt{u_0^2 + v_0^2}$. This density function is known as the *Rician density.*

(b) Show that the Rician density reduces to the Rayleigh density if $u_0 = v_0 = 0$.

MBP 7.3

$$R_o = +\sqrt{U_o^2 + V_o^2} \qquad U_o \perp\!\!\!\perp V_o \qquad \begin{array}{l} U_o \sim N(\mu_o, \sigma^2) \\ V_o \sim N(\nu_o, \sigma^2) \end{array}$$

(a) Prove that R_o is a Ricean r.v. with the pdf shown.

Let $F_{R_o}(r_o)$ denote the cdf of R_o ie

$$F_{R_o}(r_o) = Pr(R_o \leq r_o) \quad \text{where } -\infty < r_o < +\infty$$

First note that $R_o \geq 0$ owing to its definition as the positive square root given above. Thus

$$F_{R_o}(r_o) = 0 \quad \text{for} \quad r_o < 0$$

Now assuming $r_o \geq 0$. Then

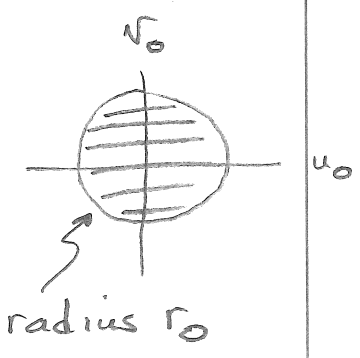
$$F_{R_o}(r_o) = Pr(R_o \leq r_o) = Pr(R_o^2 \leq r_o^2)$$

$$= Pr(U_o^2 + V_o^2 \leq r_o^2)$$

$$= \iint_{\Delta(r_o)} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(u_o-\mu_o)^2/2\sigma^2} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(v_o-\nu_o)^2/2\sigma^2} \, du_o \, dv_o$$

where

$$\Delta(r_o) = \left\{ (u_o, v_o) : \sqrt{u_o^2 + v_o^2} \leq r_o \right\}$$



$$(u_o - \mu_o)^2 + (v_o - \upsilon_o)^2$$

$$= u_o^2 - 2u_o\mu_o + \mu_o^2 + v_o^2 - 2v_o\upsilon_o + \upsilon_o^2$$

Then can write

$$F_{R_o}(r_o) = \frac{1}{2\pi\sigma^2}\iint_{\Delta(r_o)} e^{-\left[u_o^2 + v_o^2 - 2(\mu_o u_o + \upsilon_o v_o) + \mu_o^2 + \upsilon_o^2\right]/2\sigma^2} \, du_o \, dv_o$$

Now we change to polar coordinates via

$$u_o = \rho\cos\theta \qquad v_o = \rho\sin\theta$$

$$du_o\,dv_o = \rho\,d\rho\,d\theta$$

$$u_o^2 + v_o^2 = \rho^2$$

$$\mu_o u_o + \upsilon_o v_o = \mu_o\rho\cos\theta + \upsilon_o\rho\sin\theta = (\mu_o\cos\theta + \upsilon_o\sin\theta)\rho$$

$\Longrightarrow$

$$F_{R_o}(r_o) = \frac{1}{2\pi\sigma^2}\iint_{\Delta(r_o)} e^{-\left[\rho^2 - 2\rho(\mu_o\cos\theta + \upsilon_o\sin\theta) + \mu_o^2 + \upsilon_o^2\right]/2\sigma^2} \, \rho\,d\rho\,d\theta$$

<u>Note</u>: $\Delta(r_o) = \left\{(u_o, v_o) : \sqrt{u_o^2 + v_o^2} \le r_o\right\}$

$$= \left\{(\rho, \theta) : 0 \le \rho \le r_o,\ 0 \le \theta \le 2\pi\right\}$$

$$c^2 \triangleq \mu_o^2 + \upsilon_o^2$$

$$F_{R_o}(r_o) = \frac{1}{2\pi\sigma^2}\int_0^{r_o}\int_0^{2\pi} \rho\, e^{-[\rho^2 + c^2]/2\sigma^2} \, e^{\rho[\mu_o\cos\theta + \upsilon_o\sin\theta]/\sigma^2} \, d\rho\,d\theta$$

Iterating the integral

$$F_{R_0}(r_0) = \int_0^{r_0} \frac{\rho}{\sigma^2} e^{-[\rho^2+c^2]/2\sigma^2} \underbrace{\frac{1}{2\pi}\int_0^{2\pi} e^{\rho[\mu_0 \cos\theta + \upsilon_0 \sin\theta]/\sigma^2} d\theta}\, d\rho$$

Look closely at this integral which is similar to that covered in class and in MBP 7.2

$$J(\rho|\mu_0, \upsilon_0, \sigma^2) \triangleq \frac{1}{2\pi}\int_0^{2\pi} e^{\rho[\mu_0 \cos\theta + \upsilon_0 \sin\theta]/\sigma^2} d\theta$$

Following the class notes define

$$a = \rho\mu_0/\sigma^2 \qquad b = -\rho\upsilon_0/\sigma^2$$

whence

$$J = \frac{1}{2\pi}\int_0^{2\pi} e^{a\cos\theta - b\sin\theta}\, d\theta$$

$$a\cos\theta - b\sin\theta = \mathrm{Re}\left\{(a+jb)e^{j\theta}\right\}$$

Convert $a+jb = \gamma e^{j\phi}$ to polar form. Then $\gamma = \sqrt{a^2+b^2}$ and ϕ is s.t. $\gamma\cos\phi = a$ and $\gamma\sin\phi = b$

$$\Longrightarrow a\cos\theta - b\sin\theta = \sqrt{a^2+b^2}\cos(\theta+\phi)$$

$$\therefore \quad J = \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{a^2+b^2}\cos(\theta+\phi)}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{\sqrt{a^2+b^2}\cos\psi}\, d\psi \quad \text{by periodicity}$$

$$\triangleq I_0\left(\sqrt{a^2+b^2}\right)$$

$$a^2+b^2 = \rho^2(\mu_0^2+\nu_0^2)/\sigma^4$$

$$= \rho^2 c^2/\sigma^4$$

$$J = I_0\left(\frac{\rho c}{\sigma^2}\right)$$

Substituting back:

$$F_{R_0}(r_0) = \int_0^{r_0} \frac{\rho}{\sigma^2} e^{-[\rho^2+c^2]/2\sigma^2} I_0\left(\frac{\rho c}{\sigma^2}\right) d\rho$$

$\Longrightarrow$ density function for r.v. R_0 is the integrand and its known as the Ricean density

$$f_{R_0}(r_0) = \begin{cases} \dfrac{r_0}{\sigma^2} e^{-[r_0^2+c^2]/2\sigma^2} I_0\left(\dfrac{r_0 c}{\sigma^2}\right) & r_0 \geq 0 \\ 0 & r_0 < 0 \end{cases}$$

(b) Consider the zero mean case $\mu_o = \nu_o = 0$.

$\Longrightarrow c = 0$

Note that $I_o(0) = 1$ so

$$f_{R_o}(r_o) \longrightarrow \begin{cases} \dfrac{r_o}{\sigma^2} e^{-r_o^2/2\sigma^2} & r_o \geq 0 \\ 0 & r_o < 0. \end{cases}$$

which is the Rayleigh density.

7.4 Consider the receiver shown in Figure 7-6 with $w_0(t) = w_1(t) = p_T(t)$ and $\theta_i(t) = (\omega_i - \omega_c)\,t$. Assume that the LS decision rule is used. The transmitted signals are given by $s_i(t) = \sqrt{2}\,A\,a_i(t)\,p_T(t)\cos(\omega_i t + \varphi_i)$ for $i = 1$ and $i = 0$. The amplitude shaping is not the same for the two signals. The signal $a_0(t)$ is the sine pulse of duration T and $a_1(t)$ is the unit-amplitude triangular pulse of duration T; that is, $a_0(t) = \sin(\pi t/T)\,p_T(t)$ and $a_1(t) = 2t/T$ for $0 \le t \le \frac{1}{2}T$, and $a_1(t) = 2 - (2t/T)$ for $\frac{1}{2}T < t \le T$. Assume that $\omega_1 > \omega_0 \gg 1/T$, $\omega_i = 2\pi n_i/T$ for some integer n_i, and $\omega_1 - \omega_0$ is sufficiently large that, for any phase angle φ,

$$\int_0^T s_i(t)\cos(\omega_k t + \varphi)\,dt = 0$$

if $i \neq k$. The channel is an additive white Gaussian noise channel with spectral density $N_0/2$.

(a) Give a simple closed-form expression (no integrals) for the conditional error probability $P_{e,0}$ in terms of A, T, and N_0.

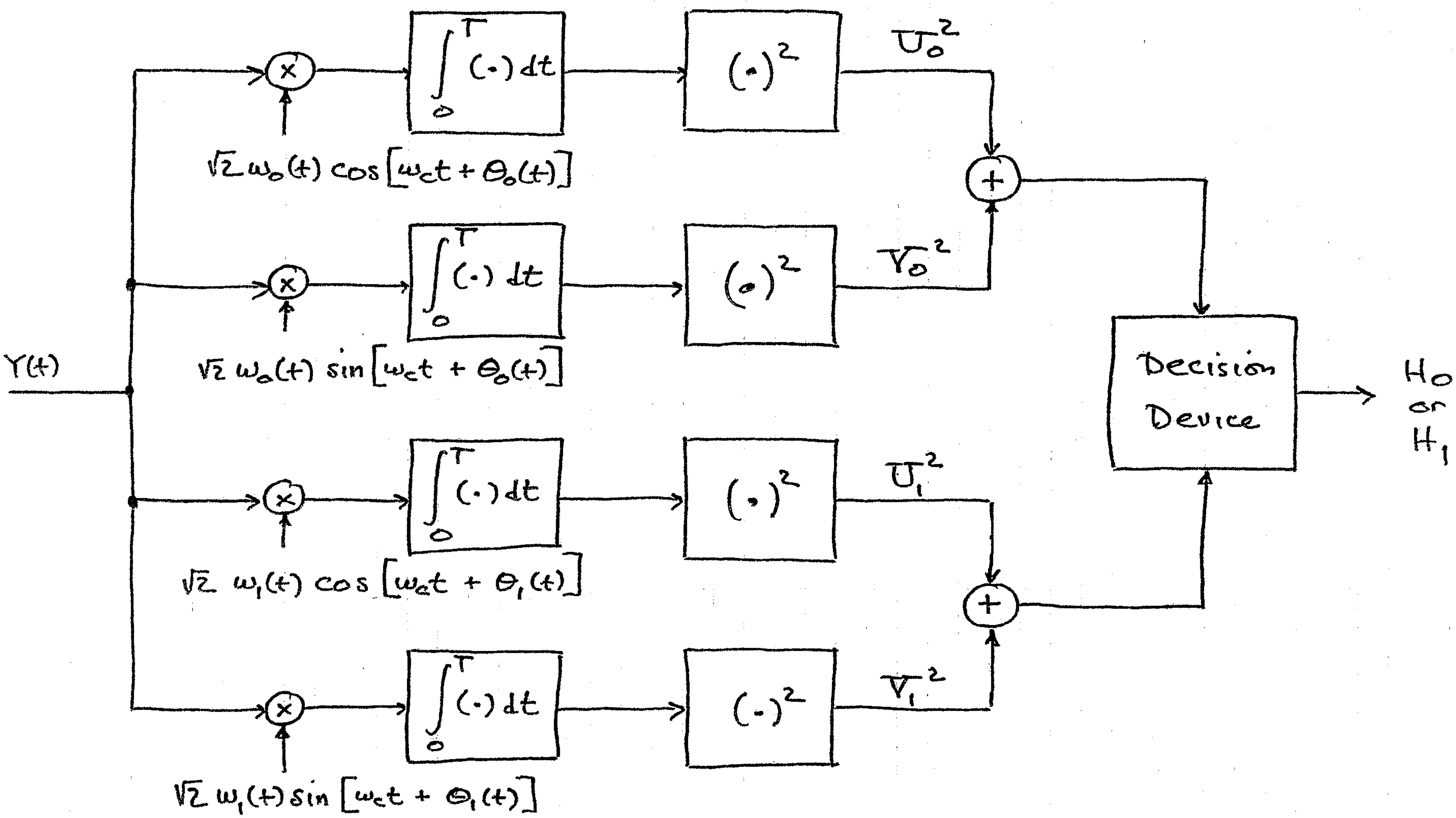
(b) Give a simple closed-form expression for $P_{e,1}$ in terms of A, T, and N_0.

(c) Are these two error probabilities the same? Should they be the same for the given system and signal set? Explain why or why not.

(d) Does the LS decision rule give a minimax decision based on the statistics R_1^2 and R_0^2? Explain your answer.

Example: MBP Prob. 7.4

Consider subopt. receiver of Fig. 7-6:



Assume: • $w_0(t) = w_1(t) = p_T(t)$; $\theta_i(t) = (\omega_i - \omega_c)t \quad i = 0,1$

• The LS (largest statistic) dr is used.

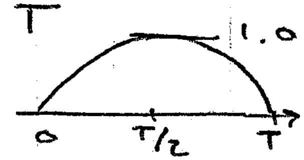
• Transmitted signals are

$$S_i(t) = \sqrt{2}\, A\, a_i(t) \cos(\omega_i t + \varphi) \qquad i = 0,1$$

where

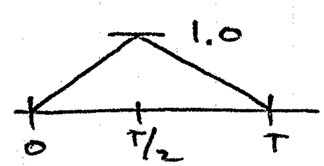
$a_0(t)$ is sine pulse of duration T

$$= \sin(\pi t/T)\, p_T(t)$$



$a_1(t)$ is unit amplitude triangular pulse of duration T

$$= \begin{cases} 2t/T & 0 \le t \le T/2 \\ 2 - 2t/T & \frac{T}{2} \le t \le T \end{cases}$$



• $\omega_1 > \omega_0 \gg 1/T$

$\omega_i = 2\pi n_i/T$ some integer n_i

• $\omega_1 - \omega_0$ is suff. large that for any phase angle φ

$$\int_0^T s_i(t) \cos(\omega_k t + \varphi)\, dt = 0 \quad \text{for } i \neq k$$

• Noice AWGN with psd height $N_0/2$

(a,b) Find $P_{e,0}$ and $P_{e,1}$ in terms of A, T, N_0.

(c) $P_{e,0} = P_{e,1}$? Should the be? Explain.

(d) Does LS dr give minimax decision based on $R_1^2 = U_1^2 + V_1^2$ and $R_0^2 = U_0^2 + V_0^2$? Explain.

Need to verify that these signal/correlator choices still fit the assumptions used in deriving the $P_{e,0}$ and $P_{e,1}$ expressions.

$$Y(t) = S_i(t) + X(t) \qquad i = 0,1$$

$$\sqrt{2}\, w_0(t) \cos\left[\omega_c t + \theta_0(t)\right] = \sqrt{2} \cos\left[\omega_0 t\right] p_T(t)$$

$$\sqrt{2}\, w_0(t) \sin\left[\omega_c t + \theta_0(t)\right] = \sqrt{2} \sin\left[\omega_0 t\right] p_T(t)$$

$$\sqrt{2}\, w_1(t) \cos\left[\omega_c t + \theta_1(t)\right] = \sqrt{2} \cos\left[\omega_1 t\right] p_T(t)$$

$$\sqrt{2}\, w_1(t) \sin\left[\omega_c t + \theta_1(t)\right] = \sqrt{2} \sin\left[\omega_1 t\right] p_T(t)$$

Correlator outputs U_0, V_0, U_1, V_1 always look like a signal component plus a noise component and we can treat these separately. First, the noise

$$X_0 = \sqrt{2} \int_0^T X(t) \cos\left[\omega_0 t\right] dt \qquad Y_0 = \sqrt{2} \int_0^T X(t) \sin\left[\omega_0 t\right] dt$$

$$X_1 = \sqrt{2} \int_0^T X(t) \cos\left[\omega_1 t\right] dt \qquad Y_1 = \sqrt{2} \int_0^T X(t) \sin\left[\omega_1 t\right] dt$$

Clearly the rvs $X_0\ Y_0\ X_1\ Y_1$ are J.G. and zero mean.

The calculation needed to compute the covariance matrix of the rvs is identical to that done in calculation of BFSK. Hence

$$\mathrm{Cov}(X_i, X_j) = E(X_i X_j) = \begin{cases} 0 & i \neq j \\ N_0T/2 & i = j \end{cases}$$

$$\mathrm{Cov}(X_i, Y_j) = E(X_i Y_j) = 0$$

$$\mathrm{Cov}(Y_i, Y_j) = E(Y_i Y_j) = \begin{cases} 0 & i \neq j \\ N_0T/2 & i = j \end{cases}$$

In otherwords these noise rvs are iid, zero mean, Gaussian with variance equal to $\sigma^2 = N_0T/2$.

The <u>only</u> assumption needed for this was that giving the form for $\omega_0(t)$, $\omega_1(t)$, $\theta_0(t)$, $\theta_1(t)$ and

$$\omega_i = 2\pi n_i / T \qquad n_i \text{ integer.}$$

Signal Parts for $i = 0$

$$s_0(t) = \sqrt{2} A\, a_0(t) \cos(\omega_0 t + \varphi)$$

$$a_0(t) \rightarrow \sin\left(\frac{\pi t}{T}\right) p_T(t)$$

$$E_0\{U_0 \mid \varphi\} = \int_0^T s_0(t) \sqrt{2} \cos(\omega_0 t)\, dt$$

$$= 2A \int_0^T \sin(\pi t / T) \cos(\omega_0 t + \varphi) \cos(\omega_0 t)\, dt$$

$$= \underbrace{A \int_0^T \sin(\pi t / T) \cos\varphi \, dt}_{\text{Term 1}} + \underbrace{A \int_0^T \sin(\pi t/T) \cos(2\omega_0 t + \varphi)\, dt}_{\text{Term 2}}$$

$$\text{Term 1} = A \cos\varphi \left[-\frac{T}{\pi} \cos\left(\frac{\pi t}{T}\right)\right]\Bigg|_0^T$$

$$= A \cos\varphi \left[1 - (-1) \right] T/\pi = \frac{2AT}{\pi} \cos\varphi$$

$$\text{Term 2} = \frac{A}{2} \int_0^T \sin\left[\left(2\omega_0 + \frac{\pi}{T}\right)t + \varphi\right] dt - \frac{A}{2} \int_0^T \sin\left[\left(2\omega_0 - \frac{\pi}{T}\right)t + \varphi\right] dt$$

$$\int_0^T \sin(\alpha t + \varphi)\, dt = -\frac{1}{\alpha}\cos(\alpha t + \varphi)\Big|_{t=0}^{T} = \frac{1}{\alpha}\Big[\cos\varphi - \cos(\alpha T + \varphi)\Big]$$

Thus

$$\text{Term } 2 = \frac{A}{2(2\omega_0 + \pi/T)}\Big[\cos\varphi - \cos(2\omega_0 T + \pi + \varphi)\Big]$$

$$- \frac{A}{2(2\omega_0 - \pi/T)}\Big[\cos\varphi - \cos(2\omega_0 T - \pi + \varphi)\Big]$$

$$\omega_0 >> 1/T \implies 2\omega_0 + \frac{\pi}{T} \approx 2\omega_0 - \pi/T \approx 2\omega_0$$

$$\therefore \text{Term } 2 \approx \frac{A}{4\omega_0}\Big\{\cancel{\cos\varphi} - \cancel{\cos\varphi} + \cos(\cancel{2\omega_0 T - \pi} + \varphi) - \cos(\cancel{2\omega_0 T + \pi + \varphi})\Big\}$$

$$= 0$$

$$E_0\{V_0|\varphi\} = \int_0^T s_0(t)\sqrt{2}\sin(\omega_0 t)\,dt$$

$$= 2A\int_0^T \sin(\pi t/T)\cos(\omega_0 t+\varphi)\sin(\omega_0 t)\,dt$$

$$= \underbrace{A\int_0^T \sin(\pi t/T)\sin(2\omega_0 t+\varphi)\,dt}_{\text{Term 1}} - \underbrace{A\int_0^T \sin(\pi t/T)\sin\varphi\,dt}_{\text{Term 2}}$$

$$\text{Term 1} = \frac{A}{2}\int_0^T \cos\left[\left(2\omega_0-\frac{\pi}{T}\right)t+\varphi\right]dt - \frac{A}{2}\int_0^T \cos\left[\left(2\omega_0+\frac{\pi}{T}\right)t+\varphi\right]dt$$

$$\int_0^T \cos(\alpha t+\varphi)\,dt = \frac{1}{\alpha}\sin(\alpha t+\varphi)\Big|_{t=0}^{T} = \frac{1}{\alpha}\left[\sin(\alpha T+\varphi)-\sin\varphi\right]$$

$$\text{Term 1} = \frac{A}{2(2\omega_0-\pi/T)}\left[\sin(2\omega_0 T-\pi+\varphi)-\sin\varphi\right]$$
$$-\frac{A}{2(2\omega_0+\pi/T)}\left[\sin(2\omega_0 T+\pi+\varphi)-\sin\varphi\right] \approx 0$$

$$\text{Term 2} = -\frac{2AT}{\pi}\sin\varphi$$

$$E_0\{U_1 \mid \varphi\} = \int_0^T s_0(t)\sqrt{2}\cos(\omega_1 t)\,dt$$

$$= 2A\int_0^T \sin(\pi t/T)\cos(\omega_0 t + \varphi)\cos(\omega_1 t)\,dt$$

$$= \underbrace{A\int_0^T \sin(\pi t/T)\cos\left[(\omega_1-\omega_0)t - \varphi\right]dt}_{\approx 0 \text{ if } \omega_1-\omega_0 \gg \pi/T} + \underbrace{A\int_0^T \sin(\pi t/T)\cos\left[(\omega_1+\omega_0)t + \varphi\right]dt}_{\approx 0}$$

for same reason as in prev. calculations

Similar argument will give

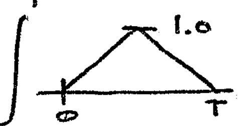
$$E_0\{V_1 \mid \varphi\} = 0.$$

<u>Signal Parts for $i=1$</u> $\quad s_1(t) = \sqrt{2}\, A\, a_1(t) \cos(\omega_1 t + \varphi)$

Can take the same approach as for $i=0$ but will be a little more tedious since will have to integrate t times a sinusoid. Under these assumptions will have

$$E_1\{U_0 | \varphi\} = E_1\{V_0 | \varphi\} \approx 0$$

$$E_1\{U_1 | \varphi\} \approx A \cos\varphi \int_0^T a_1(t)\, dt = A \cos\varphi \int_0^T$$



$$dt$$

$$= \frac{AT}{2} \cos\varphi$$

$$E_1\{V_1 | \varphi\} \approx -\frac{AT}{2} \sin\varphi$$

The setup here has been shown to meet assumptions needed in derivation of error probs for LS dr.

$$R_0^2 = U_0^2 + V_0^2 \gtrless R_1^2 = U_1^2 + V_1^2 \quad \begin{array}{l} \longrightarrow \text{decide } H_0 \\ \longrightarrow \quad \text{"} \quad H_1 \end{array}$$

with

$$\sigma^2 = N_0T/2 \qquad \alpha_0 = \frac{2AT}{\pi} \qquad \alpha_1 = \frac{AT}{2}$$

(c)

$$P_{e,0} = \frac{1}{2} e^{-\alpha_0^2/4\sigma^2} = \frac{1}{2} e^{-2A^2T/\pi^2 N_0}$$

$$P_{e,1} = \frac{1}{2} e^{-\alpha_1^2/4\sigma^2} = \frac{1}{2} e^{-A^2T/8N_0}$$

Note that $P_{e,0} \neq P_{e,1}$. In fact $\frac{2}{\pi^2} > \frac{1}{8} \Rightarrow P_{e,0} < P_{e,1}$

(d) LS dr is not an equalizer rule. Therefore not minimax.

7.6 The noncoherent communication receiver of Figure 7-4 is to be used for a packet radio communication system. Each packet consists of N binary digits that are transmitted using binary FSK modulation. The transmission rate is $1/T$ bits/s. In the absence of fading, the received signal for the nth binary digit is given by

$$s(t) = \sqrt{2}\, A \cos(\omega_i t + \varphi_i), \quad nT \leq t < (n+1)T,$$

for $i = 0$ (if 0 is sent) or $i = 1$ (if 1 is sent). Assume that $\omega_0 \neq \omega_1$ and that ω_0 and ω_1 are multiples of $2\pi/T$. The front end of the receiving radio adds white Gaussian noise $X(t)$ of spectral density $N_0/2$ to the received signal, so the input to the receiver is $Y(t) = s(t) + X(t)$.

(a) In the absence of fading, A is just a deterministic constant. What is the probability of bit error for this receiver? What is the probability of packet error for this receiver? Answer in terms of A and the other parameters given.

(b) Give a high signal-to-noise ratio approximation to the packet error probability of part (a).

(c) For a particular fading channel, the parameter A is constant over the packet, but it is a random variable that has a Gaussian distribution with mean 0 and variance β^2. Find the average bit error probability for this channel and receiver (average over the amplitudes of the received signal).

(d) In this part, the packet is deemed unacceptable if its bit error probability exceeds a specified probability p. Find an expression for the outage probability, which is the probability that the amplitude A is such that the given packet is unacceptable. Simplify as much as possible.

(e) Consider the same receiver, but a different fading channel that also distorts the pulse shape. The received signal is now of the form

$$s(t) = \sqrt{2}\, A \sin(\pi t/T) \cos(\omega_i t + \varphi_i), \quad nT \leq t < (n+1)T.$$

Assume that $T^{-1} \ll |\omega_1 - \omega_0| < \omega_0 < \omega_1$, so that, among other implications, double-frequency terms can be ignored and the signals are orthogonal. Find the average probability of bit error if the amplitude A is as described in part (c).

MBP 7.6

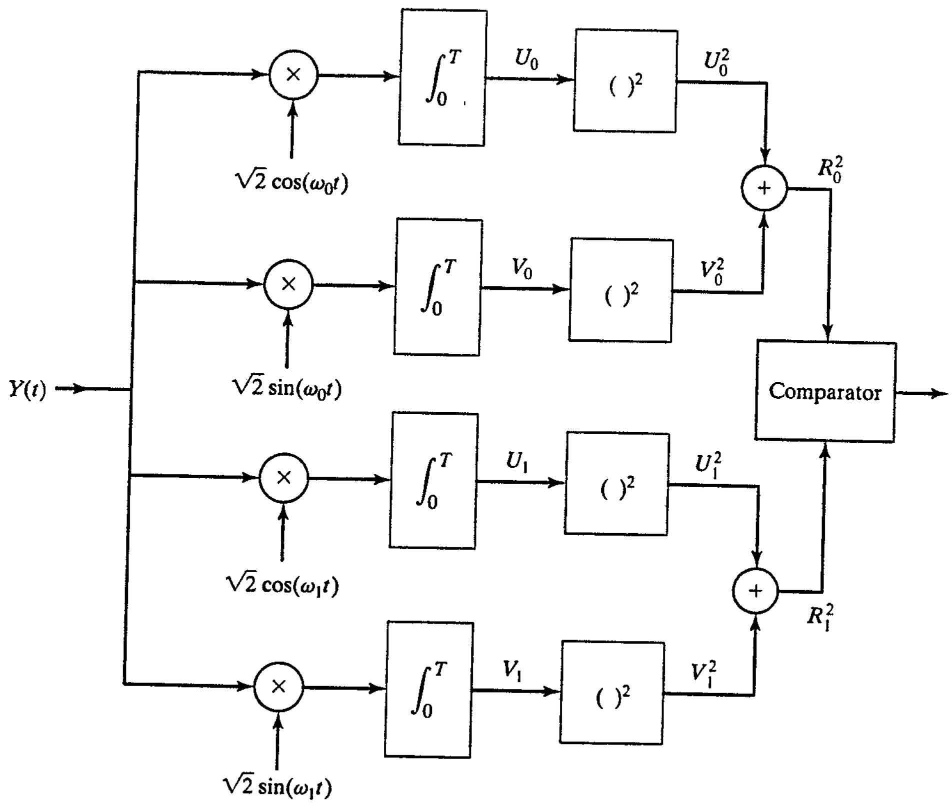


Figure 7-4: Optimum receiver for noncoherent BFSK communications.

Each packet consists of N binary digits sent via BFSK with

$$S_i(t) = \sqrt{2} A \cos(\omega_i t + \varphi_i) \qquad nT \le t < (n+1)T$$

and using the optimal non-coh. receiver arch. above. The transmission and reception of individual bits are indep.

All the standard assumptions for BFSK hold.

(a) A is a deterministic constant. Find prob. of a bit error.

We assume bits are equally likely. Prob. is also symm. in sense that

$$P_{e,0} = P_{e,1} = P_e = \frac{1}{2} e^{-A^2T/2N_0} \triangleq P_b$$

prob. of a bit error.

A packet is received correctly iff all N bits are received correctly.

$$\text{Prob of packet correct} = (1-P_b)^N$$

$$\text{Prob of packet error} = P_{e,\text{packet}} = 1-(1-P_b)^N$$

$$= 1-\left(1-\tfrac{1}{2}e^{-A^2T/2N_0}\right)^N$$

(b) If SNR is high then P_b is small. Hence

$$(1-P_b)^N \approx 1-NP_b$$

$$\Longrightarrow P_{e,\text{packet}} \approx NP_b = \frac{N}{2}e^{-A^2T/2N_0}$$

(c) Fading channel model where A is constant over a packet but varying independly from packet to packet as

$$A \sim N(0,\beta^2)$$

Want the average bit error prob. over the transmission of many packets.

$$\Longrightarrow P_b(a) = \frac{1}{2}e^{-a^2T/2N_0} = \text{conditional prob. of a bit error given } A=a$$

$$\overline{P_b} = \int_{-\infty}^{\infty} P_b(a)\, f_A(a)\, da$$

$$= \int_{-\infty}^{\infty} \frac{1}{2}e^{-a^2T/2N_0}\, e^{-a^2/2\beta^2}\frac{1}{\sqrt{2\pi}\,\beta}\, da$$

The trick for evaluating this is to recognize that the integrand is a scaled version of a Gaussian pdf.

$$\overline{P_b} = \frac{1}{2\beta}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{a^2}{2}\left(\frac{T}{N_0}+\frac{1}{\beta^2}\right)\right] da$$

Let $\lambda^2 = \left(\frac{T}{N_0}+\frac{1}{\beta^2}\right)^{-1}$ and λ the pos. sq. root.

Then

$$\overline{P_b} = \frac{\lambda}{2\beta}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\lambda} e^{-a^2/2\lambda} \, da = \frac{\lambda}{2\beta}$$

$$= \frac{1}{2\beta\sqrt{\frac{T}{N_0}+\frac{1}{\beta^2}}} = \frac{1}{2}\sqrt{\frac{N_0}{\beta^2 T + N_0}} = \frac{1}{2}\sqrt{\frac{1}{1+(\beta^2 T/N_0)}}$$

$$= \frac{1}{2}\sqrt{\frac{1}{1+\text{SNR}}} \quad \text{with } \text{SNR} \triangleq \beta^2 T/N_0$$

$$= E\{A^2\}T/N_0$$

(d) Packet said to be "unacceptable" if its bit error prob. exceeds a threshold p. It only makes sense to consider $p < 1/2$.

Define $P_b(a) = \frac{1}{2} e^{-a^2 T/2N_0}$ $\quad -\infty < a < \infty$. The outage event is then defined to be

$$\{P_b(A) > p\}$$

and the outage probability is

$$P_{out} = P\left(P_b(A) > p\right) = P\left(\frac{1}{2} e^{-A^2 T/2N_0} > p\right)$$

$$= P\left(-\frac{A^2 T}{2N_0} > \ln(2p)\right) = P\left(A^2 < \frac{2N_0}{T}\ln\left(\frac{1}{2p}\right)\right)$$

$$P_{out} = P\left(-\frac{\sqrt{\frac{2N_0}{T}\ln\left(\frac{1}{2p}\right)}}{\beta} < \frac{A}{\beta} < \frac{\sqrt{\frac{2N_0}{T}\ln\left(\frac{1}{2p}\right)}}{\beta} \right)$$

$$= \Phi\left(\frac{1}{\beta}\sqrt{\frac{2N_0}{T}\ln\left(\frac{1}{2p}\right)} \right) - \Phi\left(-\frac{1}{\beta}\sqrt{\frac{2N_0}{T}\ln\left(\frac{1}{2p}\right)} \right)$$

$$= 2\Phi\left(\frac{1}{\beta}\sqrt{\frac{2N_0}{T}\ln\left(\frac{1}{2p}\right)} \right) - 1$$

(e) Now fading channel also distorts pulse so the new received sig. model is

$$S_i(t) = \sqrt{2}\, A \sin\left(\frac{\pi t}{T}\right) \cos\left(\omega_i t + \varphi_i\right) \quad nT \le t < (n+1)T$$

We solved this as part of MBP 7.4. Found there that

$$P_b(A) = \frac{1}{2} e^{-2A^2T/\pi^2 N_0}$$

Thus in all prev. expressions we replace T by

$$T' = 4T/\pi^2$$

we will get the correct probs for this part.

$$\overline{P_b}' = \frac{1}{2}\sqrt{\frac{1}{1 + (\beta^2 4T/\pi^2 N_0)}}$$