ECE 302 Homework 6
Due July 26, 2016

Reading assignment: chapter 5, sections 5.8, 5.9; chapter 6, section 6.5 .

1. Let X and Y be random variables denoting coordinates on the xy-plane. The rotation of the point (X, Y) through θ to the point (U, V) can be performed by letting:

$$\begin{aligned} U =& X\cos\theta - Y\sin\theta \\ V =& X\sin\theta + Y\cos\theta \end{aligned}$$

Find $f_{U,V}(u, v)$.

Solution:
Problem not graded, transformation was incorrect in the assignment

(a) Density Method:

$$f_{U,V}(u, v) = f_{X,Y}(x(u, v), y(u, v)) \left| \frac{\partial(u, v)}{\partial(x, y)} \right|^{-1}$$

We have that,

$$\begin{aligned} & u = x\cos\theta - y\sin\theta,\ v = x\sin\theta + y\cos\theta \\ \implies & u\cos\theta = x\cos^2\theta - y\sin\theta\cos\theta,\ v\sin\theta = x\sin^2\theta + y\cos\theta\sin\theta \\ \implies & x = u\cos\theta + v\sin\theta. \end{aligned}$$

Similarly, $y = -u\sin\theta + v\cos\theta$.

We also have that,

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = 1.$$

Therefore,

$$f_{U,V}(u,v) = f_{X,Y}(u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta).$$

2. Two lightbulbs from different brands have lifetimes X and Y that are independent and exponentially distributed with average lifetime $1/\lambda_1$ and $1/\lambda_2$, respectively. Both lightbulbs are turned on at the same time.

   (a) Let U be the time elapsed until both lightbulbs have burned out. Find the pdf of U.

   (b) Let V be the time elapsed until the first lightbulb has burned out. Find the pdf of V.

   (c) Find the joint pdf of U and V.

Solution:

(a) U is the time elapsed until both lightbulbs have burned out, thus $U = \max\{X, Y\}$. The cdf of U can be found as follows:

$$\begin{aligned}
F_U(u) &= \Pr(U \le u) \\
&= \Pr(\max\{X,Y\} \le u) \\
&= \Pr(\{X \le u\} \cap \{Y \le u\}) \\
&= \Pr(\{X \le u\}) \Pr(\{Y \le u\}), \text{since } X,\ Y \text{ are independent} \\
&= F_X(u) F_Y(u)
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_U(u) &= \frac{d}{du} F_U(u) \\
&= \frac{d}{du} F_X(u) F_Y(u) \\
&= f_X(u) F_Y(u) + F_X(u) f_Y(u) \\
&= \lambda_1 e^{-\lambda_1 u}(1 - e^{-\lambda_2 u}) + \lambda_2 e^{-\lambda_2 u}(1 - e^{-\lambda_1 u}),\ u \ge 0
\end{aligned}$$

(b) V is the time elapsed until the first lightbulb has burned out, thus $V = \min\{X, Y\}$. The cdf of V can be found as follows:

$$\begin{aligned}
F_V(v) &= \Pr(V \leq v) \\
&= \Pr(\min\{X, Y\} \leq v) \\
&= 1 - \Pr(\min\{X, Y\} \geq v) \\
&= 1 - \Pr(\{X \geq v\} \cap \{Y \geq v\}) \\
&= 1 - \Pr(\{X \geq v\}) \Pr(\{Y \geq v\}), \text{since } X,\ Y \text{ are independent} \\
&= 1 - (1 - F_X(v))(1 - F_Y(v))
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_V(v) &= \frac{d}{dv} F_V(v) \\
&= \frac{d}{dv} 1 - (1 - F_X(v))(1 - F_Y(v)) \\
&= f_X(v)(1 - F_Y(v)) + f_Y(v)(1 - F_X(v)) \\
&= (\lambda_1 + \lambda_2) e^{-(\lambda_1+\lambda_2)v},\ v \geq 0
\end{aligned}$$

(c) Want to find the joint pdf of U and V. The joint cdf of U and V can be found by expressing the events $\{U \leq u\}$ and $\{V \leq v\}$ as $\{X \leq u\} \cap \{Y \leq u\}$ and $\{X \leq v\} \cup \{Y \leq v\}$, respectively. Therefore,

$$\begin{aligned}
F_{U,V}(u, v) &= \Pr(U \leq u \cap V \leq v) \\
&= \Pr\big((\{X \leq u\} \cap \{Y \leq u\}) \cap (\{X \leq v\} \cup \{Y \leq v\}\big) \\
&= \Pr\big((\{X \leq u\} \cap \{Y \leq u\} \cap \{X \leq v\}) \cup (\{X \leq u\} \cap \{Y \leq u\} \cap \{Y \leq v\})\big) \\
&= \begin{cases} \Pr\big((\{X \leq u\} \cap \{Y \leq u\})\big) & , v > u \\ \Pr\big((\{X \leq v\} \cap \{Y \leq u\}) \cup (\{X \leq u\} \cap \{Y \leq v\})\big) & , v \leq u \end{cases} \\
&= \begin{cases} F_{X,Y}(u, u) & , v > u \\ F_{X,Y}(v, u) + F_{X,Y}(u, v) - F_{X,Y}(v, v) & , v \leq u \end{cases}
\end{aligned}$$

Note: Can also find the joint cdf of U and V using the distribution method (see p. 235 for details on plotting areas related to min and max functions).

The joint pdf of U and V can be found as:

$$
\begin{aligned}
f_{U,V}(u,v) &= \frac{\partial^2}{\partial u \partial v} F_{U,V}(u,v) \\
&= \begin{cases} f_X(v)f_Y(u) + f_X(u)f_Y(v) & , v \le u \\ 0 & , v > u \end{cases} \\
&= \begin{cases} \lambda_1\lambda_2 \left(e^{-(\lambda_1 v + \lambda_2 u)} + e^{-(\lambda_1 u + \lambda_2 v)} \right) & , 0 \le v \le u < \infty \\ 0 & , \text{else} \end{cases}
\end{aligned}
$$

3. Let X and Y be independent Gaussian random variables with mean 0 and variance 1. Let $U = aX + bY$ and $V = cX + dY$, where $ad - bc \neq 0$.

   (a) Are U and V jointly Gaussian? Explain why.

   (b) Find the mean, variance, and correlation coefficient of U and V.

   (c) Find the joint pdf of U and V.

   (d) Find the conditional mean and variance of U given V.

   (e) Find the conditional pdf of U given V.

Solution:

(a) X and Y are independent Gaussian random variables, therefore their joint pdf is given by

$$
f_{X,Y}(x,y) = \frac{1}{2\pi} \exp\left(\frac{-(x^2+y^2)}{2} \right),
$$

which is the joint pdf of jointly Gaussian variables. From the notes, linear combinations of jointly Gaussian random variables are also jointly Gaussian random variables. Therefore, U and V are jointly Gaussian random variables.

(b)

$$
\begin{aligned}
\mu_U &= \mathbb{E}[U] \\
&= \mathbb{E}[aX + bY] \\
&= a\mathbb{E}[X] + b\mathbb{E}[Y] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\sigma_U^2 &= \operatorname{Var}[U] \\
&= \operatorname{Var}[aX + bY] \\
&= a^2 \operatorname{Var}[X] + b^2 \operatorname{Var}[Y] + 2ab \operatorname{Cov}[X, Y] \\
&= a^2 + b^2
\end{aligned}
$$

Similarly, the mean and variance of V are $\mu_V = \mathbb{E}[V] = 0$ and $\sigma_V^2 = \operatorname{Var}[V] = c^2 + d^2$.

$$
\begin{aligned}
\operatorname{Cov}[U, V] &= \operatorname{Cov}[aX + bY, cX + dY] \\
&= ac \operatorname{Cov}[X, X] + ad \operatorname{Cov}[X, Y] + bd \operatorname{Cov}[Y, X] + bd \operatorname{Cov}[Y, Y] \\
&= ac \operatorname{Var}[X] + bd \operatorname{Var}[Y] \\
&= ac + bd
\end{aligned}
$$

$$
\begin{aligned}
\rho_{UV} &= \frac{\operatorname{Cov}[U, V]}{\sqrt{\operatorname{Var}[U] \operatorname{Var}[V]}} \\
&= \frac{ac + bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}
\end{aligned}
$$

(c) Using the density method , we have that (see notes)

$$
f_{U,V}(u, v) = f_{X,Y}(x(u, v), y(u, v)) \left| \frac{\partial(u, v)}{\partial(x, y)} \right|^{-1},
$$

where

$$
x = \frac{du - bv}{ad - bc}, \; y = \frac{av - cu}{ad - bc}
$$

and

$$
\frac{\partial(u, v)}{\partial(x, y)} = ad - bc \neq 0
$$

$$
\begin{aligned}
\implies f_{U,V}(u, v) &= f_{X,Y}\left(\frac{du - bv}{ad - bc}, \frac{av - cu}{ad - bc} \right) \frac{1}{|ad - bc|} \\
&= \frac{1}{2\pi} \exp\left(-\frac{(du - bv)^2}{2(ad - bc)^2} - \frac{(av - cu)^2}{2(ad - bc)^2} \right) \frac{1}{|ad - bc|}
\end{aligned}
$$

Alternatively, since U and V are jointly Gaussian, the joint pdf of U and V can be expressed as

$$
\begin{aligned}
f_{U,V}(u,v) = &\frac{1}{2\pi\sqrt{\sigma_U^2\sigma_V^2(1-\rho_{UV})}} \\
&\exp\left(-\frac{1}{2(1-\rho_{UV})}\left(\frac{(u-\mu_U)^2}{\sigma_U^2}+\frac{(v-\mu_V)^2}{\sigma_V^2}-\frac{2\rho_{UV}(u-\mu_U)(v-\mu_V)}{\sigma_U\sigma_V}\right)\right),
\end{aligned}
$$

where μ_U, μ_V, σ_U^2, σ_V^2, and ρ_{UV} are found as in part (b).

(d) From the notes, we have that

$$
\begin{aligned}
\mu_{U|V}(v) &= \mu_U + \frac{\sigma_U}{\sigma_V}\rho_{UV}(v-\mu_V) \\
&= \frac{ac+bd}{c^2+d^2}v
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{U|V}^2 &= \sigma_U^2(1-\rho_X^2 Y) \\
&= \frac{(ad-bc)^2}{c^2+d^2}
\end{aligned}
$$

(e) From the notes we have that since U and V are jointly Gaussian then U is conditionally Gaussian given V. Therefore the conditional pdf of U given V is given by

$$
f_{U|V}(u|v) = \frac{1}{\sqrt{2\pi\sigma_{U|V}^2}}\exp\left(-\frac{(u-\mu_{U|V}(v))^2}{2\sigma_{U|V}^2}\right),
$$

where $\mu_{U|V}(v)$ and $\sigma_{U|V}^2$ are found as in part (d).

4. Let X and Y be random variables with joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} x+y & , 0 \le x \le 1, 0 \le y \le 1 \\ 0 & , \text{else} \end{cases}$$

(a) Find the MAP estimator of X given $Y = y$.

(b) Find the ML estimator of X given $Y = y$.

(c) Find the MMSE estimator of X given $Y = y$.

(d) Find the LMMSE estimator of X given $Y = y$.

Solution:

(a)

$$\begin{aligned} f_Y(y) &= y + \frac{1}{2},\ 0 \le y \le 1 \\ f_{X|Y}(x|y) &= \frac{x+y}{y+\frac{1}{2}},\ 0 \le x, y \le 1 \end{aligned}$$

The MAP estimator is given by

$$\begin{aligned} \widehat{X}_{MAP}(y) &= \arg\max_x f_{X|Y}(x|y) \\ &= \arg\max_x \frac{x+y}{y+\frac{1}{2}},\ 0 \le x, y \le 1 \\ &= 1,\ 0 \le y \le 1 \end{aligned}$$

Therefore, $\widehat{X}_{MAP}(y) = 1$.

(b)

$$\begin{aligned} f_X(x) &= x + \frac{1}{2},\ 0 \le x \le 1 \\ f_{Y|X}(y|x) &= \frac{x+y}{x+\frac{1}{2}},\ 0 \le x, y \le 1 \end{aligned}$$

The ML estimator is given by

$$\begin{aligned} \widehat{X}_{ML}(y) &= \arg\max_x f_{Y|X}(y|x) \\ &= \arg\max_x \frac{x+y}{x+\frac{1}{2}},\ 0 \le x, y \le 1 \end{aligned}$$

$f_{Y|X}(y|x)$ has no critical points as a function of x within $[0,1]$, therefore the maximum must occur at the endpoints. Since, $f_{Y|X}(y|0) = 2y$ and $f_{Y|X}(y|1) = 2/3y + 2/3$, $\widehat{X}_{ML}(y)$ is given by

$$\widehat{X}_{ML}(y) = \begin{cases} \frac{2}{3}(y+1) & , 0 \leq y < 1/2 \\ 2y & , 1/2 \leq y \leq 1 \end{cases}$$

(c) The MMSE estimator is given by

$$\begin{aligned} \widehat{X}_{MSE}(y) &= \mathbb{E}[X|Y=y] \\ &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx \\ &= \frac{1}{y+\frac{1}{2}} \int_0^1 (x^2 + xy)dx \\ &= \frac{3y+2}{6y+3}, \; 0 \leq y \leq 1 \end{aligned}$$

(d) The LMMSE estimator is given by

$$\widehat{X}_{LMMSE}(y) = \mu_X + \frac{\sigma_X}{\sigma_Y}\rho_{XY}(y - \mu_Y) \quad .$$

The mean, variance, and correlation coefficient of X and Y are : $\mu_X = \mu_Y = 7/12$, $\sigma_X^2 = \sigma_Y^2 = 11/144$, $\rho_{XY} = -1/11$. Therefore,

$$\widehat{X}_{LMMSE}(y) = \frac{7}{12} - \frac{1}{11}(y - \frac{7}{12}), 0 \leq y \leq 1$$