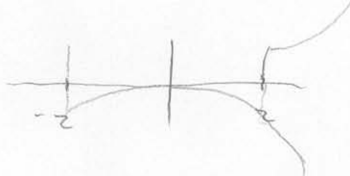

(15 pts) **1.** Compute the Fourier transform of the DT signal

$$x[n] = n^2u[n-2] - n^2u[n+2]$$

(Express your answer as a linear combination of sine and/or cosine functions.)

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(n^2 u[n-2] - n^2 u[n+2]\right) e^{-j\omega n}$$

$$= \sum_{n=-2}^{2} n^2 e^{-j\omega n} = 4e^{j2\omega} + e^{j\omega} + 0 + e^{-j\omega} + 4e^{-j2\omega}$$

$$= 4e^{j2\omega} + 4e^{-j2\omega} + e^{j\omega} + e^{-j\omega}$$

$$\boxed{= 8\cos(2\omega) + 2\cos(\omega)}$$

(15 pts) **2.** Show that the Fourier transform of the CT signal $x(t) = cos(\omega_0 t)$ is $\mathcal{X}(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$.

$\rightarrow$ $\mathcal{F}(x(t))$ is ~~too~~ hard so

$$\mathcal{F}^{-1}(\mathcal{X}(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{X}(\omega) e^{j\omega t}\, d\omega \qquad (2)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)\right) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left(\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right) e^{j\omega t}$$

$$= \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$, by the sifting property

$$= \boxed{\cos(\omega_0 t) = x(t)}$$

(odd)^2 ⇒ even

(15 pts) **3.** Given is a DT signal $x[n] = \frac{1}{g[n]^2}$ where $g[n]$ is a pure imaginary signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{cos\omega}$. Explain why Bob's answer is wrong.

↳ pure imaginary and even

x[n] is real and even

∴ X(ω) must be real and even (35)

→←

∴ X(ω) should be real

b) Alice says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{1}{sin\omega}$. Could Alice be right? Explain.

x[n] is real and even

∴ X(ω) must be real and even (35)

→←

X(ω) is odd

No.

c) Devin says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{1}{\omega^2}$. Could Devin be right? Explain.

X(ω) is not periodic and the F.T. of a D.T. signal is ALWAYS periodic.

→←

Devin is wrong.

4. A discrete-time LTI system has frequency response

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}.$$

(15 pts) a) Derive a difference equation relating the input and the output of this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

$$Y(\omega) = X(\omega)\,H(\omega) \quad \Rightarrow \quad \frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{2}{\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right)}$$

$$= \left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right)Y(\omega) = 2X(\omega)$$

(25)

$$\Rightarrow \quad x[n-n_0] \xrightarrow{\mathcal{F}} e^{-j\omega n_0}X(\omega)$$

$$\Rightarrow Y(\omega) - \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-2j\omega}Y(\omega) = 2X(\omega)$$, by linearity (24)

$$\boxed{\Rightarrow y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]}$$

(10 pts) b) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$?

$$\mathcal{F}(x[n]) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}} \quad , \text{by } (42)$$

$$Y(\omega) = X(\omega)\, H(\omega)$$

$$\boxed{= \frac{2}{\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right)} \cdot \frac{1}{1 - \frac{1}{4}e^{-j\omega}}}$$

(15 pts) b) Find the unit impulse response of this system.

$x[n] = \delta[n]$

$\mathcal{F}(\delta[n]) = 1$

$$y(w) = \left(\frac{2}{\left(1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-2jw}\right)}\right)\frac{8}{8} = \frac{2 \cdot 16}{\left(8 - 6e^{-jw} + (e^{-jw})^2\right)} = \frac{16}{(e^{-jw}-2)(e^{-jw}-4)}$$

$$= \frac{A}{(e^{-jw}-2)} + \frac{B}{(e^{-jw}-4)} = \frac{-8}{e^{-jw}-2} + \frac{8}{e^{-jw}-4}$$

$$= \frac{4}{1 - \frac{1}{2}e^{-jw}} + \frac{-2}{1 - \frac{1}{4}e^{-jw}}$$

$$\xrightarrow{\mathcal{F}^{-1}} \boxed{4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]}$$, by (24) and (42)

(20 pts) **5.** Use the definition of the Fourier transform (*not* the properties listed in the table) to prove the following Fourier transform property.

$$x(at+b) \xrightarrow{F} \frac{e^{j\omega\frac{b}{a}}}{-a} \mathcal{X}\left(\frac{\omega}{a}\right) \text{ for any } a, b \text{ real numbers with } a < 0.$$

$$\mathcal{F}(x(at+b)) = \int_{-\infty}^{\infty} x(at+b)\, e^{-j\omega t}\, dt$$

let $\tau = at+b$

$\therefore t = \frac{\tau - b}{a}$

$\therefore dt = \frac{d\tau}{a}$

$$= \int_{\infty}^{-\infty} x(\tau)\, e^{-j\omega\left(\frac{\tau}{a} - \frac{b}{a}\right)} \frac{d\tau}{a} = \frac{e^{j\omega\frac{b}{a}}}{-a} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\left(\frac{\tau}{a}\right)}\, d\tau$$

let $\lambda = \frac{\omega}{a}$

$$= \frac{e^{j\omega\frac{b}{a}}}{-a} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\lambda\tau}\, d\tau$$

$$= \frac{e^{j\omega\frac{b}{a}}}{-a} X(\lambda)$$ Substitute for λ

$$\boxed{= \frac{e^{j\omega\frac{b}{a}}}{-a} X\left(\frac{\omega}{a}\right)}$$