1. Let $f : [0,1] \cup [2,3] \to \mathbb{R}$ continuous. If the image of f is connected, show f is not 1-1.

2. Let A and B be subsets of a metric space X.

   (a) Recall $d(x, A) := \inf_A d(x, a)$. If A is compact, show there is some $a \in A$ where the distance is obtained.

   (b) Suppose $X = \mathbb{R}^n$ and A is only assumed to be closed. Prove the result still holds.

   (c) Find a counter-example to show this is false in general when A is assumed only to be closed.

   (d) Now define $d(A, B) := \inf \{d(a, b) : a \in A,\ b \in B\}$. Show that if A and B are both compact there are $a \in A, b \in B$ for which the distance is obtained.

   (e) Can we relax this condition?

3. Lebesgue Number Lemma

   Let X be a metric space, and K a compact subset. Fix $\{G_\alpha\}$ a given open cover, and show there exists some $\delta > 0$ such that for every $k \in K, B_\delta(k) \subset G_\alpha$ for some α.

   (Hint: WLOG $\{G_\alpha\} = \{G_1 \ldots G_N\}$define $f(x) = \sum_1^N d(x, G_j^c))$.

4. Let X be a metric space. Say a subset K is "sequentially compact" $\iff$ for ever sequence $\{x_n\} \subset K$ there is a subsequence $\{x_{n_k}\}$ which converges in K.

   Prove K is compact $\iff$ K is sequentially compact.

5. Let X be a metric space. Say a subset K is "totally bounded" $\iff$ for every $\epsilon > 0$ there are $x_1, ..., x_n \in K$ such that $\cup_{j=1}^n N_\epsilon(x_j) \supset K$.

   Show K is compact $\iff$ K is closed and totally bounded

6. Show bounded need not imply totally bounded in a metric space and conclude that the Heine-Borel property does not hold in general. (Recall, a metric space X satisfies the Heine-Borel property $\iff$ closed and bounded is equivalent to compact for subsets of X.)