(15 pts) 1. Using the definition of the Fourier transform, compute the Fourier transform of the DT signal:

$$x[n] = e^{j\frac{\pi}{17}n}\left(\frac{1}{3}\right)^n u[n-1].$$

Frequency shifting

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad e^{j\omega_0 n}x[n] \rightarrow X e^{j(\omega-\omega_0)}$$

So Let $x[n] = \left(\frac{1}{3}\right)^n u[n-1]$ and use frequency shift

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$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n u[n-1]\, e^{-j\omega n}$$

$$u[n-1] = 0 \qquad n-1<0$$
$$n<1$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n e^{-j\omega n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{3}e^{-j\omega}\right)^n \qquad \text{let } m = n-1, \quad n = m+1$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{3}e^{-j\omega}\right)^{m+1} \Rightarrow \frac{1}{3}e^{-j\omega}\sum_{m=0}^{\infty}\left(\frac{1}{3}e^{-j\omega}\right)^m$$

$$X(\omega) = \frac{1}{3}e^{-j\omega}\cdot\frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

then frequency shift by $\frac{\pi}{17}$

$$X(\omega) = \frac{1}{3}e^{-j(\omega-\pi/17)}\,\frac{1}{1-\frac{1}{3}e^{j\left(\frac{\pi}{17}-\omega\right)}}$$

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You could compare your answer with the one you get without using the freq. shift. property

(15 pts) 2. Using the definition of the inverse Fourier transform, compute the CT inverse Fourier transform of

$$X(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \delta(\omega - k\pi).$$

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$$\mathcal{F}^{-1} = \frac{1}{2\pi} \int_0^{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \delta(\omega - k\pi) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{j\omega n} \underbrace{\sum_{k=0}^{\infty} \frac{1}{2^k} \delta(\omega - k\pi)}_{\text{depends on } k} d\omega$$

$$= \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{1}{2^k} \int_0^{2\pi} e^{j\omega n} \delta(\omega - k\pi) d\omega)$$

$$x[n] = \boxed{\frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{1}{2^k} e^{jk\pi n}} \quad \text{by "sifting property"}$$

$$= \frac{1}{1 - e^{j\pi t}/2}$$

$x[n]$ is purely imaginary and ev

(~~cos ... ... [...]~~)

(15 pts) **3.** Given is a DT signal $x[n] = j\cos(g[n])$ where $g[n]$ is a real signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $X(\omega) = \frac{j}{\sin\omega}$. Explain why Bob's answer is wrong.

~~$X(\omega)$ is odd and~~

Test if $X(\omega)$ is real or imag.

If real, $X(\omega) = X^*(-\omega) = \frac{-j}{\sin -\omega} = \frac{-j}{-\sin\omega} = \frac{j}{\sin\omega}$

which indicates $x[n]$ is real, but $j\cos(g[n])$ is imaginary

NO

b) Alice says that the Fourier transform of $x[n]$ is $X(\omega) = \frac{3}{\cos\omega}$. Could Alice be right? Explain.

$X^*(-\omega) = \frac{3}{\cos -\omega} = \frac{3}{\cos\omega}$ ~~so $x[n]$ would be real,~~ but $j\cos(g[n])$ is NOT, so Alice is not right

c) Devin says that the Fourier transform of $x[n]$ is $X(\omega) = \frac{j}{(\omega^2+1)^2}$. Could Devin be right? Explain.

$X^*(-\omega) = \frac{-j}{((-\omega)^2+1)^2} = \frac{-j}{(\omega^2+1)^2} \neq X(\omega)$ so $x[n]$ is not r

So might be correct

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4. A discrete-time LTI system is defined by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

(10 pts) a) Find the frequency response of this system. (Use the properties of the Fourier transform listed in the table to justify your answer. Do not just plug into a formula you have learned by heart.)

$$x[n-n_0] \xrightarrow{\mathcal{F}} e^{-j\omega n_0} X(\omega)$$

$$Y(\omega) - \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-2j\omega}Y(\omega) = 2X(\omega)$$

$$Y(\omega)\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right) = 2X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \boxed{\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}}}$$

(10 pts) b) What is the unit impulse response of this system. (Justify your answer)

$h[n] = \mathcal{F}^{-1}(H(\omega))$

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$$\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

$$a^n u[n] \xrightarrow{\mathcal{F}} \frac{1}{1-ae^{-j\omega}}$$

$$\boxed{h[n] = 4\cdot\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]}$$

(5 pts) c) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$? (Justify your answer.)

$y[n] = x[n] * h[n]$

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or

$Y(\omega) = X(\omega)\cdot H(\omega)$

$$X(\omega) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$Y(\omega) = X(\omega)\cdot H(\omega) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}\cdot\frac{2}{\left(1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}\right)}$$

can stop here

$$= \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)}\cdot\frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$\boxed{Y(\omega) = \frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2\left(1-\frac{1}{2}e^{-j\omega}\right)}}$$

(10 pts) d) What is the output when the input is $x[n] = (\frac{1}{4})^n u[n]$? (Justify your answer)

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$\mathcal{F}^{-1}(Y(\omega)) \Rightarrow \frac{A}{(1-\frac{1}{4}e^{-j\omega})} + \frac{B}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{C}{(1-\ldots}$?

$x = e^{-j\omega}$

$\frac{2}{(1-\frac{1}{4}x)^2(1-\frac{1}{2}x)} = \frac{A}{1-\frac{1}{4}x} + \frac{B}{(1-\frac{1}{4}x)^2} + \frac{C}{(1-\frac{1}{2}x)}$ on scratch

$B = -2 \quad C = 8 \quad A = -4$

$\boxed{y[n] = -4\left(\frac{1}{4}\right)^n u[n] - 2(n+1)\left(\frac{1}{4}\right)^n u[n] + 8(}$

Be careful when writing your answer. It needs to be clear, clean, and logical

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(15 pts) **5.** Use the properties of the Fourier transform to evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{\sin^2(4t)}{t^2}\,dt.$$

Use property 20:

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{\sin(4t)}{t}\right|^2 dt \quad \text{need} \quad \mathcal{F}\left(\frac{\sin(4t)}{t}\right)$$

using FS: $\frac{\sin Wt}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} u(\omega+W) - u(\omega-W)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{\pi \sin(4t)}{\pi t}\right|^2 \quad ??$$

stuck

Make the sequence of steps clear