(15 pts) **1.** Using the definition of the Fourier transform, compute the Fourier transform of the DT signal:

$$x[n] = e^{j\frac{\pi}{4}n}\left(\frac{1}{3}\right)^n u[n-1].$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n u[n-1]\, e^{j(\frac{\pi}{4}-\omega)n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{3} e^{j(\frac{\pi}{4}-\omega)}\right)^n u[n-1] \qquad x = n-1$$

$$= \frac{1}{3} e^{j(\frac{\pi}{4}-\omega)} \sum_{x=-\infty}^{\infty} \left(\frac{1}{3} e^{j(\frac{\pi}{4}-\omega)}\right)^x u[x]$$

$\downarrow$ By (3.1) $\qquad \left|\frac{1}{3} e^{j(\frac{\pi}{4}-\omega)}\right| < 1$

$$= \frac{1}{3} e^{j(\frac{\pi}{4}-\omega)} \cdot \frac{1}{1-\frac{1}{3} e^{j(\frac{\pi}{4}-\omega)}}$$

$$\left(X(\omega) = \frac{e^{j(\frac{\pi}{4}-\omega)}}{3 - e^{j(\frac{\pi}{4}-\omega)}} \right)$$

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(15 pts) 2. Using the definition of the inverse Fourier transform, compute the CT inverse Fourier transform of

$$X(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \delta(\omega - k\pi).$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \delta(\omega - k\pi) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \underbrace{\int_{-\infty}^{\infty} \delta(\omega - k\pi) e^{j\omega t} d\omega}_{= e^{jk\pi t}}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] e^{jk\pi t}$$

$\omega = k$

$$= \frac{1}{2\pi} \sum_{k=0}^{\infty} \left(\frac{1}{2} e^{j\pi t}\right)^k$$

$$= \frac{1}{2\pi} \cdot \frac{1}{1 - \frac{1}{2} e^{j\pi t}}$$

$$\boxed{x(t) = \frac{1}{2\pi - \pi e^{j\pi t}}}$$

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(15 pts) 3. Given is a DT signal $x[n] = j\cos(g[n])$ where $g[n]$ is a real signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $X(\omega) = \frac{j}{\sin\omega}$. Explain why Bob's answer is wrong.

$X(\omega) = \frac{j}{\sin(\omega)}$ is odd and pure imaginary.
By (4.5), $x[n]$ must be real and odd.
But $x[n] = j\cos(g[n])$ is pure imaginary.
Therefore, $X(\omega) = \frac{j}{\sin\omega}$ cannot be its transform

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b) Alice says that the Fourier transform of $x[n]$ is $X(\omega) = \frac{2}{\cos\omega}$. Could Alice be right? Explain.

$X(\omega) = \frac{2}{\cos(\omega)}$ is real and even. This means that $x[n]$ must be real and even by (4.4). But $x[n]$ is imaginary! So $X(\omega)$ cannot be $\frac{2}{\cos\omega}$

5

c) Devin says that the Fourier transform of $x[n]$ is $X(\omega) = \frac{j}{\omega^2+1}$. Could Devin be right? Explain.

$X(\omega) = \frac{j}{(\omega^2+1)}$ is not periodic. It is known that the F.T. of a discrete time signal is always periodic.

Thus, $X(\omega) = \frac{j}{(\omega^2+1)}$ cannot be the transform of $x[n] = j\cos(g[n])$

5

nicely worded

4. A discrete-time LTI system is defined by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

(10 pts) a) Find the frequency response of this system. (Use the properties of the Fourier transform listed in the table to justify your answer. Do not just plug into a formula you have learned by heart.)

$$\mathcal{F}\left(y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2]\right) = 2\,\mathcal{F}(x[n])$$

$$Y(\omega) - \frac{3}{4}e^{-j\omega}Y(\omega) + \frac{1}{8}e^{-j2\omega}Y(\omega) = 2X(\omega)$$

$$Y(\omega)\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}\left(e^{-j\omega}\right)^2\right)$$

$z = e^{-j\omega}$

$\frac{1}{8}(8 - 6z + z^2)$

$\frac{1}{8}(8 - 6z + z^2)$

$\frac{1}{8}(2-z)(4-z)$

$$Y(\omega)\,(2 - e^{-j\omega})(4 - e^{-j\omega})\cdot\frac{1}{8} = 2X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \boxed{\frac{16}{(2 - e^{-j\omega})(4 - e^{-j\omega})}} = H(\omega)$$

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no need to factor

0

(10 pts) b) What is the unit impulse response of this system. (Justify your answer)

$\mathcal{F}(SYS[\delta[n]]) = H(\omega) \cdot \mathcal{F}(\delta[n])$

? ? ?

$= H(\omega) \cdot 1$

$h_{\text{response}} = \mathcal{F}^{-1}(\mathcal{F}(SYS[\delta[n]])) = \mathcal{F}^{-1}(H(\omega))$

(5 pts) c) What is the Fourier transform of the output when the input is $x[n] = (\frac{1}{4})^n u[n]$? (Justify your answer.)

$X(\omega) = \mathcal{F}\left((\frac{1}{4})^n u[n]\right)$

$X(\omega) = \dfrac{1}{1 - \frac{1}{4}e^{-j\omega}}$

$Y(\omega) = \mathcal{F}(y[n]) = H(\omega) X(\omega)$

$= \dfrac{16}{(2 - e^{-j\omega})(4 - e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$

$Y(\omega) = \dfrac{4}{(2 - e^{-j\omega})(4 - e^{-j\omega})(4 - e^{-j\omega})}$

5

(10 pts) d) What is the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$? (Justify your answer)

Partial Equations!

If you explained a bit more, you might get some partial credit.

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(15 pts) **5.** Use the properties of the Fourier transform to evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{\sin^2(4t)}{t^2}\,dt.$$

$$= \int_{-\infty}^{\infty} \pi^2 \left|\frac{\sin 4t}{\pi t}\right|^2 dt$$

$$= \pi^2 \int_{-\infty}^{\infty} \left|\frac{\sin 4t}{\pi t}\right|^2 dt$$

Parseval!

$$= \pi^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\mathcal{F}\left(\frac{\sin 4t}{\pi t}\right)\right|^2 d\omega$$

$$= \frac{\pi}{2} \int_{-\infty}^{\infty} \left| u(\omega+4) - u(\omega-4)\right|^2 d\omega$$

$$= \frac{\pi}{2} \int_{-4}^{4} 1\, d\omega$$

$$= \frac{\pi}{2} \cdot 8$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin^2(4t)}{t^2}\,dt = 4\pi}$$

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