10 APRIL

Graph Coloring

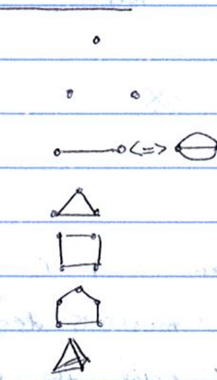
Recall: G is bipartite exactly if one can decorate each vertex either red or blue such that adjacent vertices have different colors
$\Leftrightarrow$ G does not have simple ciruit of odd length

Def A coloring of G using n colors is a way of labeling the vertices of G by $\{1, \dots, n\}$ such that labels of adjucent vertices are different

Why color?
e.g. Political Maps (coloring countries to display borders)
$\Leftrightarrow$ planar graphs

Def: χ_G, chromatic number, is the minimum number of colors needed to fi a coloring

ex



χ_G	G
1	
1	
2	
3	
2	
3	
4	
n	K_n
2	$K_{m,n}$
$\infty, -1$	

χ_G ↗

We note:
→ χ_G makes no sense if G has a loop
→ χ_G is not affected if G has multiple edges
→ disconnected G can be resolved by observing connected subgraphs

For coloring purpose, we assume G to be simplet connected

For Maps:
$\Rightarrow$ G will be planar
$\Rightarrow$ what is the min # colors?

Mapmakers (cartographers?)

= "4 is enough..."

WANT: a general bound on χ_G that works for all simple connected planar graphs

4-Color - Conjecture : A Kenpe ~1860

$\chi_G \leq 4$ for all simple planar graphs

Approaching the Conjecture...

Lemma

Every simple and connected and planar graphs contains at least one vertex of degree 5 or less.

Prf:

Last time... simple planar connected graph:

$$e \leq 3v - 6$$

Suppose Lemma is false i.e. degree ≥ 6.

By Handshake Lemma

$$2e = \sum_{vert} \text{degree} \geq \sum_{vert} 6$$

$$\Rightarrow 2e \geq 6v$$

$$e \geq 3v \quad \times$$

Proposition I

Any planar, simple, (connected) G can be 6-colored.

↳ Find vertex with at most 5 degree (gauranteed by Lemma).

Erase this vertex, 6 color the rest of its neighbors and color the ~~remaining~~ vertex w/ remaining color

Proposition II:

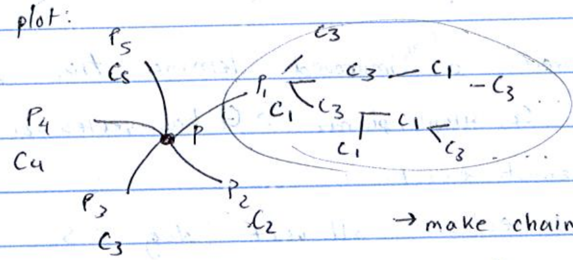
5 colors are enough!

Idea: Find vertex of degree no more than 5

$= 5$ ↙ ↘ < 5

erase, color the rest, color vertex w/ color not used by neighbor

→ plot:



→ make chains of $C_1 - C_3$ on P_1.

2 Things can happen

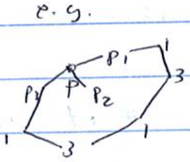
(a) the $C_1 - C_3$ chain emanating from P eventually touches P_3

(b) If it does not touch P_3 within the chain, flip-flop the coloring of C_1 & C_3 so that what was colored C_1 is now C_3 and vice versa. This does not affect labeling

Then, P_1, colored C_3, does not use C_1 any longer. we only needed 5 colors.

In case (a), the 1-3 chains sprouting from b, touches P_3

e.g.



then, this chain forms a circuit

Consider P_2 that lies ~~inside~~ circumscribed by the circuit.

Repeat the idea with 2-4 chain:

Build 2-4 chain of ~~P~~ sprouting from P_2. By diagram & planarity, this chain cannot connect to P_4 without violating def of planar graph.

⇒ can flip-flop C_2 on P_2 ... eliminate the need of extra color!

Q The argument had nothing to do w/ P_5 or C_5.
Can we do the same to prove 4-color theorem.

NO:
we would need a improved lemma that would have to say "G, simple, planar $\Rightarrow$ G has vertex of degree ≤ 4"
such lemma cannot exist...
e.g. icosahedron: all vert deg = 5.

HISTORY
~1974 CS Researchers proposed a proof:
→ There is a finite list of L graphs such that if one checks each graph in L to be 4-colorable, then proof of 4-colorability of all simple planar graph can be accepted

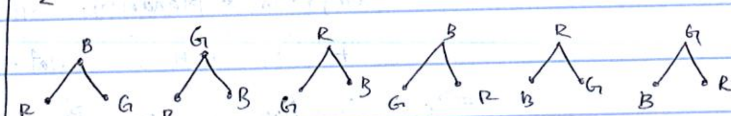
→ FAILURE: too many errors, too long list of graphs
too long code (>3000 pgs) too long proof (>5000)
and wrong

~1989 Robin Thomas
Distilled the proof from 3000 → 50 pgs
Reduced List to ~ 30 graphs.

Def For any graph G, let $\chi_G(n)$ = # of coloring of G using n colors

ex.  = G

n	$\chi_G(n)$
0	0
1	0
2	2
3	



+ 2 colors = R−G R−B B−G.

= 12

Ex a) $G = \Delta$

n	$\chi_G(n)$
3	$6 = 3!$
4	$\binom{4}{3} 3!$
5	$\binom{5}{3} 3!$

b) $G = K_n$

$0, \ldots, n-1 \longrightarrow 0$

if $m > n \rightarrow \binom{m}{n} \cdot n! = \frac{m!}{(n!)(m-n)!} \cdot n!$

$= \frac{m!}{(m-n)!} = \underline{m(m-1)\ldots(m-n+1)}$.

to be cont. Thursday.