Assignment 4 - Normal Familes & Rouché's theorem

1. Does there exist a family of polynomials $Q_n(z)$ converging uniformly to $\frac{1}{z}$ on $|z| = 1$?

2. Suppose $f(z)$ is entire and define

$$\mathcal{F} = \{f(kz) \mid k \in \mathbb{C}\}.$$

Show that $\mathcal{F}$ is normal if and only if f is a polynomial.

3. Let f be analytic on the open unit disk. Show that $\sum_{n=1}^{\infty} f(z^n)$ converges uniformly on compact $K \subset \mathbb{D}$.

4. Determine the number of zeroes of

$$f(z) = 2 - z^3 - e^{-z}$$

on the upper half plane $\{\operatorname{Im} z > 0\}$.

5. Show that if $Q(z)$ is a polynomial, there exists z with $|z| = 1$ such that

$$|Q(z) - \frac{1}{z}| \geq 1.$$

6. Show that all roots of $f(z) = z^6 - 5z^2 + 10$ lie in the annulus $\{1 < |z| < 2\}$.

7. Let $f_n : \mathbb{D} \to \mathbb{C}$ be a sequence of injective holomorphic functions, and suppose $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. Show that f is either injective or constant.

8. Define

$$\Pi = \{x + iy : |x| < \frac{\pi}{4}, -\infty < y < \infty\},$$

and suppose $f \in \mathcal{O}(\Pi)$ satisfies

$$|f(z)| \leq 1, \quad f(0) = 0.$$

Show that $|f(z)| \leq |\tan z|$.

9. Let $\mathcal{F}$ denote the set of all holomorphic mappings of the unit disk to itself satisfying $f(\frac{1}{2}) = 0$. Find

$$\sup_{\mathcal{F}} \{\operatorname{Im} f(0)\}.$$

10. (a) State Rouché's theorem.
    (b) Use Rouché's theorem to prove the Fundamental Theorem of Algebra, i.e. that a polynomial $Q(z)$ of degree n has exactly n zeroes counted with multiplicity.