MA598 - Complex Analysis Qual Prep - Summer 2014
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Assignment 2

6/26/2014

1. (a) State Morera's theorem.
   (b) Use Morera's theorem to show that if f is holomorphic in
   $$\mathcal{O}(\mathbb{C} - \{|z| = 1\})$$
   and continuous for $|z| = 1$, then f is entire.

2. Suppose $f \in \mathcal{O}(\mathbb{D})$, and that f extends continuously to the boundary with $|f(z)| = 1$ for $|z| = 1$.
   (a) Show that f maps $\mathbb{D}$ to itself.
   (b) Show that f is constant if it has no zeroes.
   (c) Show that f may be written
   $$f(z) = \zeta \prod_{i=1}^{n} \frac{z - a}{1 - \bar{a}z}$$
   for some ζ on the unit circle.
   (d) Show that if f satisfies the above conditions, the Taylor series for f about $z = 0$ has radius of convergence $R > 1$. Does this remain true if the condition of continuity on the unit circle is dropped?

3. Define $f(z) = \int_0^1 \frac{dt}{1+tz}$.
   (a) Use Morera's theorem to show that $f \in \mathcal{O}(\mathbb{D})$.
   (b) Find a power series expansion for $f(z)$ on $\mathbb{D}$.

4. Let C_1 denote the unit circle parametrized in the usual sense. Compute
   $$\int_{C_1} \frac{dz}{z^2 + z - \sigma}$$
   where σ is a real number satisfying $0 < \sigma < 2$.

5. Convert the integral
   $$\int_0^{2\pi} \frac{d\theta}{2 + \sin\theta}$$
   into a contour integral of the the form $\int_C f(z)dz$ where C is the unit circle and f is a rational function, and use the Residue Theorem to compute the integral.

6. Suppose f is an analytic function mapping the unit disc into itself with two distinct fixed points. Show that $f(z) = z$.