12/4. Problems for chap. 10.

10.1 – 10.15 \ 10.5. 10.12

10.21 – 10.27 10.30 – 10.33. 10.43 – 10.44

z - transform

z - transform is the DT analogue of the L.T

Recall : $z^n \longrightarrow \boxed{LTI} \longrightarrow H(z) z^n$

where $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$ } z-transform of h[n].

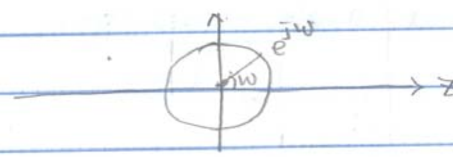
Def: Let x[n] be a DT signal . The z-transform of x[n] is $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

where z is a complex variable. $Z(x[n])$

* Relationship between z-transform and F.T.

① $X(w) = X(e^{jw})$

↳ F.T ↳ z-transform



because $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$] F.T. formula

② write $X(z) = X(re^{jw})$, then $X(re^{jw})$ is the F.T of $x[n] r^{-n}$

because $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$X(re^{jw}) = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-jwn}$ F.T of $x[n] r^{-n}$ $r \in [0, \infty)$

* Note : z-transform sometimes diverge

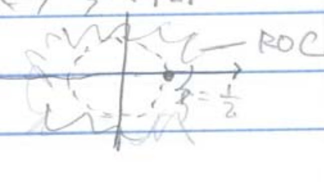
Def: The set of complex number z such that $\sum_{n=-\infty}^{\infty} x[n] z^{-n}$ converges is called the "region of convergence" (ROC) of the z-transform

Ex: $x[n] = u[n] / 2^n$

$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$= \sum_{n=-\infty}^{\infty} \frac{u[n]}{2^n} z^{-n} = \sum_{n=0}^{\infty} \frac{z^{-n}}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n = \begin{cases} \text{diverges, else} \\ \frac{1}{1-\frac{1}{2z}} \text{ if } \left|\frac{1}{2z}\right| < 1 \end{cases}$

$\left|\frac{1}{2z}\right| < 1 \iff \frac{|1|}{|2||z|} < 1 \iff \frac{1}{2|z|} < 1 \iff \frac{1}{2} < |z|$



Ex: $x[n] = \dfrac{-u[-n-1]}{2^n}$

$$X(z) = \sum_{n=-\infty}^{\infty} \frac{-u[-n-1]}{2^n} z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -\frac{z^{-n}}{2^n} \qquad k=-n$$

$$= \sum_{k=1}^{\infty} -\frac{z^k}{2^{-k}}$$

$$= -\sum_{k=1}^{\infty} (2z)^k$$

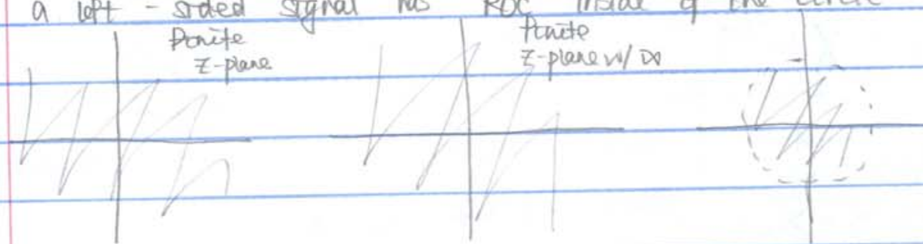
$$= -\left(\sum_{k=0}^{\infty} (2z)^k - (2z)^0\right)$$

$$= \begin{cases} \text{diverges}, & \text{if } |2z| \geq 1 \\ -\dfrac{1}{1-2z} + 1, & \text{if } |2z| < 1 \end{cases}$$

$$\iff |z| < \tfrac{1}{2}$$

* a right-sided signal has ROC outside of the circle

* a left-sided signal has ROC inside of the circle



$X(z) = \dfrac{P(z)}{Q(z)}$ The ROC does not contain any pole.

If $x[n]$ is of finite duration, then ROC is either $\begin{cases} \text{entire } z\text{-plane} \\ \text{entire } z\text{-plane except } z=0 \\ \text{entire } z\text{-plane except } z=0 \ \& \ z=\infty \\ \text{entire } z\text{-plane except } z=\infty \end{cases}$

Ex: $x[n] = \delta[n]$

$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n}$

$= 1 \cdot z^{0}$

$= 1 \cdot 1 = 1$ $\therefore$ ROC contains all complex numbers z including $z=0$

because $X(0) = \sum \delta[n] 0^{-n} = 1 \cdot 0^{-0} = 1 \cdot 0^{0} = 1$

to look at $z=\infty$ look at $X(\frac{1}{z})$ at $z=0$

$X(\frac{1}{z}) = 1$ is well defined at $z=0$

$\Longrightarrow$ $z=\infty$ is in the ROC

if $x[n] = \delta[n+1]$

$X(z) = \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n}$

$= 1 \cdot z^{(-1)}$

$= z$

$\therefore$ all z-plane is in the ROC

$X(0) = 0$ is well defined

$\Rightarrow$ $z=0$ is also in ROC

$X(\frac{1}{z}) = \frac{1}{z}\Big|_{z=0}$ diverges $\Rightarrow$ $z=\infty$ is not in ROC

Example of taking inverse z-transform

say $X(z) = 3z^{-1} + 5z^{-2}$

observe $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$= \cdots + x[-1] z^{1} + x[0] z^{0} + x[1] z^{-1} + x[2] z^{-2} + \cdots$

$x[1] = 3$ $\quad$ $x[2] = 5$

$x[n] = 0$ when $n \neq 1, 2$

Now say $X(z) = \frac{1}{1-2z}$, $\quad |z| < \frac{1}{2}$

$= \sum_{n=0}^{\infty} (2z)^{n}$ $\qquad$ Let $k = -n$.

$= \sum_{n=0}^{\infty} 2^{n} z^{n} = \sum_{k=-\infty}^{0} 2^{-k} z^{-k}$

the coeff. of z^{-k} in $\qquad$ is equal to $x[k]$

when $k > 0$, $x[k] = 0$

when $k \leq 0$ $\quad x[k] = 2^{-k}$.

or $x[k] = 2^{-k} u[-k]$

or $x[n] = 2^{-n} u[-n]$

$\frac{1}{1-2z} = \frac{1}{2z(\frac{1}{2z} - 1)} = \left(\frac{-1}{2z}\right) \frac{1}{1 - \frac{1}{2z}}$ $\qquad |z| > \frac{1}{2}$ $\Rightarrow \frac{1}{2|z|} < 1$.

$= \frac{-1}{2z} \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^{n} = -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \frac{1}{z^{n+1}}$