<u>Ex</u>: Let X, Y be r.v.s with jpdf

$$f_{X,Y}(x,y) = 2e^{-(x+y)} \quad , 0 \le y \le x < \infty$$

Find $\hat{X}_{MSE}(y)$ and $\hat{X}_{LMMSE}(y)$ given $Y=y$.

$$\hat{X}_{MSE}(y) = E[X \mid Y=y]$$

$$= \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y)\, dx$$

$$= \int_{y}^{\infty} x \, e^{-(x-y)} \, dx$$

$$= e^{y} \int_{y}^{\infty} x e^{-x} \, dx$$

$$= e^{y}\left(e^{-y}(y+1)\right)$$

$$\boxed{= y+1}$$

$$\hat{X}_{LMMSE}(y) = \rho_{XY} \frac{\sigma_X}{\sigma_Y}(y-\mu_Y) + \mu_X$$

$$\mu_X = 3/2 \ , \quad \mu_Y = 1/2 \ , \quad \sigma_X = \sqrt{5}/2 \ , \quad \sigma_Y = 1/2$$

$$\rho_{XY} = 1/\sqrt{5}$$

$$\hat{X}_{LMMSE}(y) = \frac{1}{\cancel{\sqrt{5}}} \cdot \frac{\cancel{\sqrt{5}/2}^{\,1}}{1/2}(y - 1/2) + 3/2$$

$$\boxed{= y+1}$$

Note: $\hat{X}_{MSE}(y) \neq \hat{X}_{LMMSE}$ in general.

If X, Y are jointly Gaussian $\Rightarrow \hat{X}_{MSE}(y) = \hat{X}_{LMMSE}(y)$

If $\hat{X}_{MSE}(y)$ is a linear function of y

$$\Leftrightarrow \hat{X}_{MSE}(y) = \hat{X}_{LMMSE}(y)$$

Random Processes

Consider a random experiment with sample space S.

An <u>outcome</u> is a possible result of a random experiment.

A <u>random variable</u> (r.v.) X assigns a number to each outcome.

A <u>random process</u> (r.p.) $X(t)$ assigns a time function (or waveform) to each outcome.

Typically the time index t belongs to an index set T, where T can describe continuous or discrete time values.

<u>Ex</u>: Continuous T:

$$T = (-\infty, \infty) \ , \quad [0, \infty) \ , \quad [a, b]$$

Discrete T:

$$T = \{\ldots, -1, 0, 1, \ldots\} \ , \quad \{0, 1, 2, \ldots\},$$

$$\{n, n+1, \ldots, m\}$$

For <u>fixed</u> time values $t_1, \dots, t_N \in T$,

$X(t_1), \dots, X(t_n)$ are r.v.s

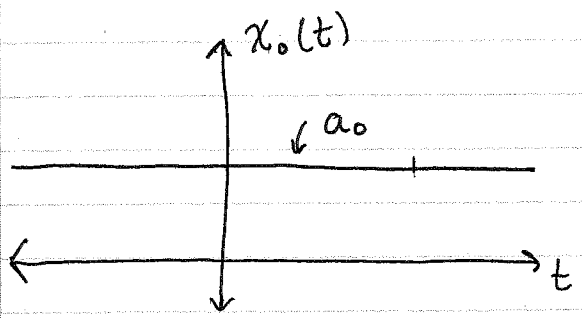
For <u>fixed</u> outcomes, $X(t)$ is a deterministic function $x(t)$. This function is called the <u>sample function</u> corresponding to an outcome $s \in S$.

<u>Ex</u>: 1) Constant r.p.

$$X(t) = A, \quad -\infty < t < \infty,$$

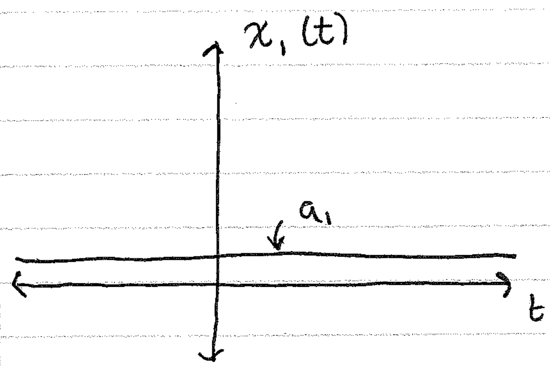
where A is ~~an~~ a r.v. with pdf $f_A(a)$

Sample functions of $X(t)$ lock like



$A = a_0$

$X(t) = x_0(t) = a_0$



$A = a_1$

$X(t) = x_1(t) = a_1$

Fix $t_0 \in (-\infty, \infty)$

$X(t_0)$ is a r.v. with pdf

$$f_{X(t_0)}(x) = f_A(x)$$

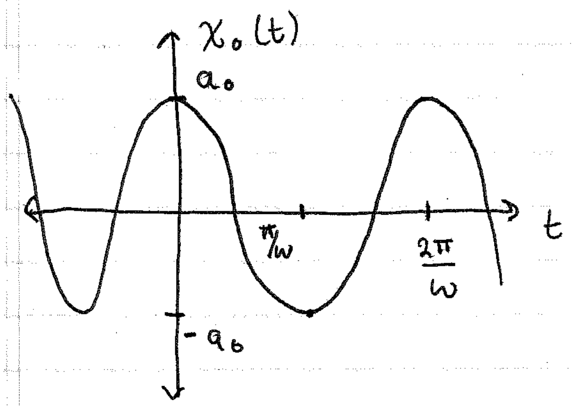
2) Cosine with random amplitude

$$X(t) = A\cos(\omega t) \quad , \quad -\infty < t < \infty$$

where A is a r.v. with pdf $f_A(a)$

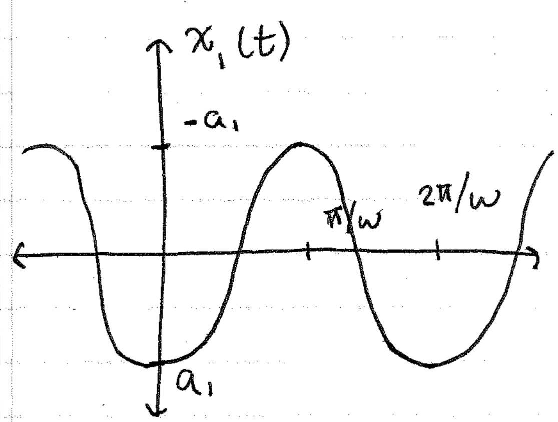
and ω is a positive constant.

Sample functions of $X(t)$ look like



$A = a_0 > 0$

$X(t) = x_0(t) = a_0 \cos(\omega t)$



$A = a_1 < 0$

$X(t) = x_1(t) = a_1 \cos(\omega t)$

Fix $t_0 \in (-\infty, \infty)$ such that $\cos(w t_0) = 0$

$X(t_0)$ is a r.v., $X(t_0) = A \cos(w t_0)$

$$f_{X(t_0)}(x) = f_A\left(\frac{x}{\cos(wt_0)}\right) \cdot \frac{1}{|\cos(wt_0)|}$$

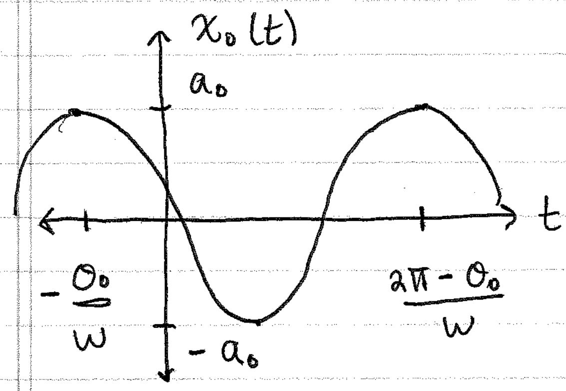
3) Cosine with random amplitude and random phase.

$$X(t) = A\cos(wt + \Theta), \quad -\infty < t < \infty$$

where A, Θ are r.v.s with jpdf $f_{A,\Theta}(a,\theta)$

and w is a positive function.

Sample functions of $X(t)$ look like



$A = a_0$, $\Theta = \theta_0$

$X(t) = x_0(t) = a_0 \cos(wt + \theta_0)$

Fix $t_0 \in (-\infty, \infty)$

$X(t_0)$ is a r.v., $X(t_0) = A\cos(wt_0 + \Theta)$

Can find $f_{X(t_0)}(x)$ from $f_{A,\Theta}(a,\theta)$ and the relationship $X(t_0) = A\cos(wt_0 + \Theta)$