

# ECE 438 HW1 Solution

$$\begin{aligned}
 1. \quad X(w) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\
 &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t} dt \\
 &= \left. \frac{1}{-j\omega} e^{-j\omega t} \right|_{-\frac{1}{2}}^{\frac{1}{2}} \\
 &= \frac{1}{-j\omega} \left( e^{-j\omega \frac{1}{2}} - e^{-j\omega (-\frac{1}{2})} \right) \\
 &= \frac{1}{\omega} \frac{1}{j} \left( e^{j\omega \frac{1}{2}} - e^{-j\omega \frac{1}{2}} \right) \\
 &= \frac{2}{\omega} \sin \frac{\omega}{2} \\
 &= \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}}
 \end{aligned}$$

$\omega = 2\pi f$

$$X(f) = \frac{\sin \pi f}{\pi f}$$

$$2. \quad \text{If } X(t) \leftrightarrow X(\omega)$$

$$\text{then } X(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$\text{Proof: } F\{X(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt$$

$$\text{Let } \tau = at. \quad t = \tau/a$$

$$\text{then } F\{X(at)\} = \begin{cases} \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j(\omega/a) \cdot \tau} d\tau & a > 0 \\ -\frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j(\omega/a) \cdot \tau} d\tau & a < 0 \end{cases}$$

$$\text{therefore, } F\{X(at)\} = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$3. X(t) = \operatorname{sinc}(3t)$$

$$\text{Let } X_1(t) = \operatorname{sinc}(t)$$

$$= \frac{\sin \pi t}{\pi t}$$

Using duality property. if  $\operatorname{rect}(t) \longleftrightarrow \operatorname{sinc}(f)$

then  $\operatorname{sinc}(t) \longleftrightarrow \operatorname{rect}(-f) = \operatorname{rect}(f)$

$$\therefore X_1(f) = \operatorname{rect}(f)$$

Using time scaling property. if  $\operatorname{sinc}(t) \longleftrightarrow \operatorname{rect}(f)$ ,

$$\text{then } \operatorname{sinc}(3t) \longleftrightarrow \frac{1}{3} \operatorname{rect}\left(\frac{f}{3}\right)$$

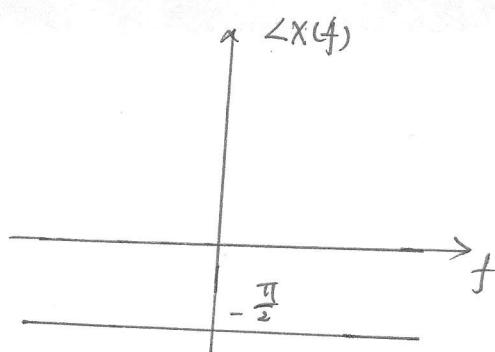
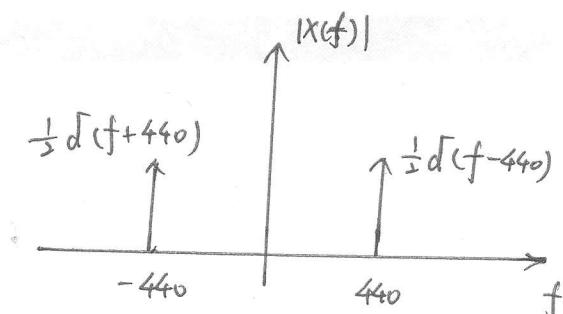
$$\text{Therefore. } \operatorname{sinc}(3t) \longleftrightarrow \frac{1}{3} \operatorname{rect}\left(\frac{f}{3}\right)$$

$$4. X(t) = \operatorname{sinc}(880\pi t) = \frac{1}{2j} (e^{j880\pi t} - e^{-j880\pi t})$$

$$\text{Because } e^{j2\pi \cdot 440 t} \longleftrightarrow \operatorname{rect}(f - 440)$$

$$e^{j2\pi (-440)t} \longleftrightarrow \operatorname{rect}(f + 440)$$

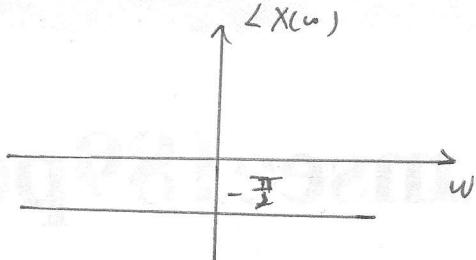
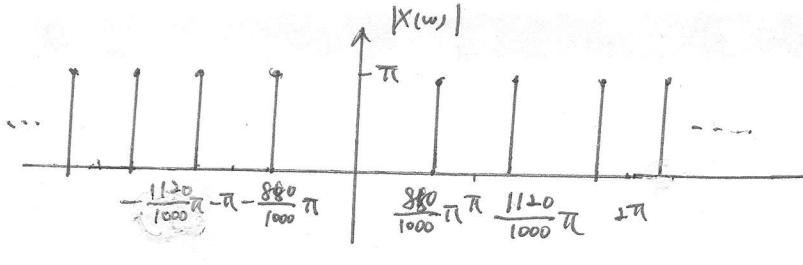
$$\Rightarrow X(t) \longleftrightarrow \frac{1}{2j} (\operatorname{rect}(f - 440) - \operatorname{rect}(f + 440))$$



$$5. \quad x_1[n] = \sin\left(\frac{880\pi n}{1000}\right)$$

$$= \frac{1}{2j} \left( e^{j\frac{880}{1000}\pi n} - e^{-j\frac{880}{1000}\pi n} \right)$$

$$\longleftrightarrow \frac{1}{2j} \left( 2\pi \operatorname{rep}_{2\pi} d(w - \frac{880\pi}{1000}) - 2\pi \operatorname{rep}_{2\pi} d(w + \frac{880\pi}{1000}) \right)$$

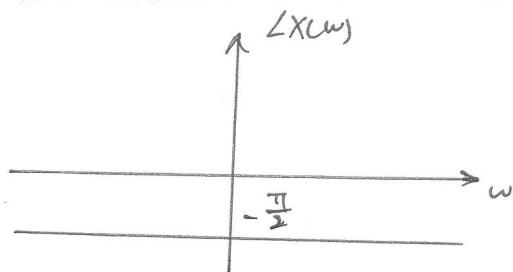
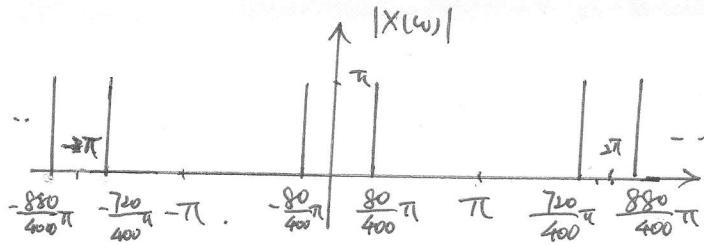


$$x_2[n] = \sin\left(\frac{880\pi}{400}n\right)$$

$$= \frac{1}{2j} \left( e^{j\frac{880}{400}\pi n} + e^{-j\frac{880}{400}\pi n} \right)$$

$$= \frac{1}{2j} \left( e^{j\frac{80}{400}\pi n + j2\pi n} + e^{-j\frac{80}{400}\pi n - j2\pi n} \right)$$

$$\longleftrightarrow \frac{1}{2j} \left( 2\pi \operatorname{rep}_{2\pi} d(w - \frac{80}{400}\pi) - 2\pi \operatorname{rep}_{2\pi} d(w + \frac{80}{400}\pi) \right)$$



Compared with 4. CTFT., DTFT in frequency domain is periodic.

6. Let  $\alpha = \frac{1}{2}$   $\beta = 3$   $a = 1$ ,  $b = 0$ ,  $c = -1$ ,  $d = -1$

$$\text{then } x[n] = \left(\frac{1}{2}\right)^n u[n] + (3)^n u[-n-1]$$

$$\begin{aligned} a) X(\omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] e^{-j\omega n} + \sum_{n=-\infty}^{\infty} (3)^n u[-n-1] e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} (3)^n e^{-j\omega n} \\ &= \frac{1}{1 - \frac{1}{2}e^{-j\omega}} + \sum_{k=1}^{\infty} (3)^{-k} e^{jk\omega} \quad (k = -n) \\ &= \frac{1}{1 - \frac{1}{2}e^{-j\omega}} + \frac{\frac{1}{3}e^{j\omega}}{1 - \frac{1}{3}e^{j\omega}} \\ X(\omega) &= \frac{1}{1 - \frac{1}{2}e^{-j\omega}} - \frac{1}{1 - 3e^{-j\omega}} \end{aligned}$$

$$b) x[n] = \left(\frac{1}{2}\right)^n u[n] + (3)^n u[-n-1]$$

$$\Rightarrow X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 - 3z^{-1}} \quad \frac{1}{2} < |z| < 3$$

c) replace  $z = e^{j\omega}$  b) will be equal to a)

$$7.a) \quad X_1(z) = \frac{1}{1 + 2z} \quad |z| < 2$$

$$= \frac{\frac{1}{2}z^{-1}}{1 + \frac{1}{2}z^{-1}}$$

We know that  $\frac{1}{1 + \frac{1}{2}z^{-1}} \leftrightarrow \begin{cases} (-\frac{1}{2})^n u[n] & -\frac{1}{2} < |z| < 2 \\ -(-\frac{1}{2})^n u[-n-1] & |z| < \frac{1}{2} \end{cases}$

$$\frac{z^{-1}}{1 + \frac{1}{2}z^{-1}} \leftrightarrow \begin{cases} (-\frac{1}{2})^{n-1} u[n-1] & \frac{1}{2} < |z| < 2 \\ -(-\frac{1}{2})^{n-1} u[-n] & |z| < \frac{1}{2} \end{cases}$$

Time shifting

$$\frac{\frac{1}{2}z^{-1}}{1 + \frac{1}{2}z^{-1}} \leftrightarrow \begin{cases} -(-\frac{1}{2})^n u[-n-1] & \frac{1}{2} < |z| < 2 \\ (-\frac{1}{2})^n u[-n] & |z| < \frac{1}{2} \end{cases}$$

Therefore,  $X_1(z) \leftrightarrow \begin{cases} -(-\frac{1}{2})^n u[n-1] & -\frac{1}{2} < |z| < 2 \\ (+\frac{1}{2})^n u[-n] & |z| < \frac{1}{2} \end{cases}$

b)  $X_2(z) = \frac{1}{1 - \frac{1}{2}z} = \frac{-2z^{-1}}{1 - 2z^{-1}} \quad |z| < 2$

$$\frac{1}{1 - 2z^{-1}} \leftrightarrow - (2)^n u[-n-1]$$

$$\frac{z^{-1}}{1 - 2z^{-1}} \leftrightarrow - (2)^{n-1} u[-n]$$

$$\frac{-2z^{-1}}{1 - 2z^{-1}} \leftrightarrow (2)^n u[-n]$$

Therefore  $X_2(z) \leftrightarrow 2^n u[-n] \quad |z| < 2$

$$(c) X_3(z) = \frac{1}{z^2 + 2z - 3} = \frac{1}{(z-1)(z+3)} \quad |z| < 3$$

Let  $X_3(z) = \frac{A}{z-1} + \frac{B}{z+3}$

$$= \frac{(A+B)z + (3A-B)}{(z-1)(z+3)}$$

$$\begin{cases} A+B=0 \\ 3A-B=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \end{cases}$$

$$X_3(z) = \frac{\frac{1}{4}}{z-1} + \frac{-\frac{1}{4}}{z+3}$$

$$= \frac{\frac{1}{4}z^{-1}}{1-z^{-1}} + \frac{-\frac{1}{4}z^{-1}}{1+3z^{-1}}$$

$$\longleftrightarrow \frac{1}{4}u[n-1] + \frac{1}{4}(-3)^{n-1}u[n]$$

$$(d) X_4(z) = \ln(1+z)$$

$$nX_4[n] \longleftrightarrow -z \frac{dX_4(z)}{dz}$$

$$-z \frac{dX_4(z)}{dz} = -\frac{z}{1+z} = -\frac{1}{1+z^{-1}} \quad |z| < 1$$

$$-\frac{1}{1+z^{-1}} \longleftrightarrow (-1)^n u[-n-1]$$

$$\text{Therefore } nX_4[n] = (-1)^n u[-n-1]$$

$$\therefore X_4[n] = \frac{(-1)^n}{n} u[-n-1]$$