## HOMEWORK 5

## REQUIRED PROBLEMS:

I. Canonical Form Problems: 7, 8
II. Observability Problems: 4,5
III. Controllability Problems: 4,5
IV. Observability and Controllability Problems: 5, 6
V. Theoretical Problems: 3, 4, 5, 6

## I. CANONICAL FORM Problems

1. (2nd order ccf) (a) Find the controllable canonical form of the scalar differential equation below.

$$
D^{2} y+4 D y-y=-u+9 D u+2 D^{2} u
$$

(b) If the initial conditions on the differential equation are $\mathrm{y}(0)=3, \dot{y}(0)=0$, and $\mathrm{u}(\mathrm{t})=$
$(1-\mathrm{t}) \cos (\mathrm{t})$, determine the proper initial conditions on the controllable canonical state model as follows:
(i) Derive the necessary equations to be solved in terms of state model matrices, A, B, C, and D, i.e. use the symbols $\mathrm{A}, \mathrm{B}, \mathrm{C}$, and D where A is $2 \times 2$, B is $2 \times 1$, etc; do not use the matrices computed in part (a) yet.
(ii) Using the matrices computed in part (a), find the proper $\mathrm{x}(0)$.
(c) The parameters used to compute the initial condition obtained in (b) were corrupted by noise and luckily you discovered the true desired initial condition for the state model as $\mathrm{x}\left(0^{+}\right)=[3,-1]^{\mathrm{T}}$.
Assuming that $\mathrm{x}\left(0^{-}\right)=[1,-3]^{\mathrm{T}}$, find an impulsive input which will set up these initial conditions on your state model.
(d) Find the transfer function of the system, $H(s)=\frac{y(s)}{u(s)}$. Given the A, B, C, and D matrices of your state model, show that it is also true that $H(s)=C(s I-A)^{-1} B+D$.
2. (2nd order ocf) (a) Repeat problem 1, using the observable canonical form. Note that even though you have new $\mathrm{A}, \mathrm{B}, \mathrm{C}$, and D matrices, the transfer function is the same.
(b) Is the "impulsive" input the same? Explain why or why not.
3. (2nd order ccf) (a) Find the controllable canonical state model realization of the differential equation $\ddot{y}+a_{1} \dot{y}+a_{2} y=b_{2} u+b_{1} \dot{u}+b_{0} \ddot{u}$.
(b) If $a_{1}=2, a_{2}=-2, b_{2}=1, b_{1}=-1, b_{0}=1$ and $\dot{y}\left(\pi^{-}\right)=\mathrm{y}\left(\pi^{-}\right)=1$ and $\mathrm{u}(\mathrm{t})=\sin (\mathrm{t}) 1^{+}(\mathrm{t})$, find the initial condition on this state model consistent with the ICs and input on the differential equation.
(c) Find an impulsive input which will set up this initial condition at $\mathrm{x}\left(\pi^{+}\right)$given that $\mathrm{x}\left(\pi^{-}\right)=0$.
4. (2nd order ocf) Repeat problem 4 for the observable canonical form. Is the "impulsive" input the same? Explain why or why not.
5. (2nd order ocf) The observable canonical form of a particular differential equation whose initial conditons are $\mathrm{y}(1)=\dot{y}(1)=1$ when $\mathrm{u}(\mathrm{t})=\mathrm{t} 1^{+}(\mathrm{t})$ is given by

$$
\begin{aligned}
& {\left[\begin{array}{l}
\dot{x}_{1}(t) \\
\dot{x}_{2}(t)
\end{array}\right]=\left[\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]+\left[\begin{array}{c}
1 \\
-1
\end{array}\right] u(t)} \\
& y(t)=\left[\begin{array}{ll}
-1 & 1
\end{array}\right] x(t)+[1] u(t)
\end{aligned}
$$

(a) Find $x(1)$.
(b) Because noise corrupted the measurements of $y(1)$ and $\dot{y}(1)$, you had to recompute the initial state vector only to find the proper initial conditon to be $\mathrm{x}(1)=\left[\begin{array}{cc}5 & -2\end{array}\right]^{\mathrm{T}}$. Assuming that $\mathrm{x}\left(1^{-}\right)=\left[\begin{array}{ll}0 & 0\end{array}\right]^{\mathrm{T}}$, compute the structure and the coefficients of an impulsive input which will set up this initial condition on the system.
6. (3rd order ccf) The controllable canonical form of a particular system is

$$
\left[\begin{array}{l}
\dot{x}_{1} \\
\dot{x}_{2} \\
\dot{x}_{3}
\end{array}\right]=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & 1
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]+\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right] u y=\left[\begin{array}{lll}
1 & -1 & 1
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]+[1] u
$$

(a) Suppose $\mathrm{u}(\mathrm{t})=\mathrm{e}^{\mathrm{t}}$ for $\mathrm{t} \geq-1$ and $\mathrm{y}(0)=5, \dot{\mathrm{y}}(0)=10$, and $\ddot{\mathrm{y}}(0)=12$. Find the value of $\mathrm{x}(\mathrm{t})$ at $\mathrm{t}=0$.
(b) Now suppose $x\left(0^{-}\right)=\left[\begin{array}{lll}1 & 1 & 1\end{array}\right]^{\mathrm{T}}$ and you desire it to be $\mathrm{x}\left(0^{+}\right)=\left[\begin{array}{lll}0 & 0 & 0\end{array}\right]^{\mathrm{T}}$. Construct an impulsive input that will set up $x\left(0^{+}\right)$.
7. (3rd order ocf) A scalar third order differential equation with initial conditions, $y(0)=\dot{y}(0)=\ddot{y}(0)$ $=1$, is driven by the input $u(t)=\sin (t)$. After much blood, sweat, tears, and coffee, you have found that the observable canonical form is

$$
\left[\begin{array}{l}
\dot{x}_{1} \\
\dot{x}_{2} \\
\dot{x}_{3}
\end{array}\right]=\left[\begin{array}{ccc}
0 & 0 & 1 \\
1 & 0 & -1 \\
0 & 1 & -1
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]+\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right] u \quad y=\left[\begin{array}{lll}
0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right]+[1] u
$$

(a) Determine $\mathrm{x}(0)=\left[\mathrm{x}_{1}(0), \mathrm{x}_{2}(0), \mathrm{x}_{3}(0)\right]^{\mathrm{T}}$.
(b) Find an impulsive input of the form $u(t)=\xi_{0} \delta(t)+\xi_{1} \dot{\delta}(t)+\xi_{2} \ddot{\delta}(t)$ that will set up the initial condition $\mathrm{x}\left(0^{+}\right)=\left[\begin{array}{lll}1 & 2 & 3\end{array}\right]^{\mathrm{T}}$ on the state model assuming $\mathrm{x}\left(0^{-}\right)=0$. Note that this initial condition should be different from your answer in part a.
8. (3rd order ccf) Let D be the derivative operator.
(a) Derive, as per class development, a state model in the controllable canonical form for the differential equation

$$
D^{3} y+a_{1} D^{2} y+a_{2} D y+a_{3} y=b_{3} u+b_{2} D u+b_{1} D^{2} u+b_{0} D^{3} u
$$

(b) If $a_{1}=-1, a_{2}=a_{3}=0, b_{0}=1, b_{1}=-1, b_{2}=1, b_{3}=1, D^{2} y(1)=2, \operatorname{Dy}(1)=1, y(1)=2.6788$, and $u(t)$ $=10 t \exp (-t) 1^{+}(t)$, determine $x(1)$ so that the response for $t \geq 1$ of the state model and the differential equation will coincide.
(c) Determine an impulsive input which will set up the initial condition, $\mathrm{x}\left(1^{+}\right)$, computed in (b) assuming $x\left(1^{-}\right)=0$. Precisely specify the structure of the impulsive input.
(d) Find state feedback which will place the poles of the system at $\{0,-1,-2\}$.
9. (4th order ocf) (a) Determine the observable canonical form of the differential equation

$$
D^{4} y-y=u-b_{3} D u-0.5 D^{2} u+2 D^{3} u+D^{4} u
$$

(b) If $D^{3} y(1)=D^{2} y(1)=D^{1} y(1)=y(1)=1, u(t)=10 t \exp (-t) 1^{+}(t)$, and $b_{3}=0.5$, determine $x(1)$ so that the response for $t \geq 1$ of the state model and the differential equation will coincide.
(c) If possible, determine an impulsive input which will set up the initial condition, $\mathrm{x}\left(1^{+}\right)$, computed in (b) assuming $x\left(1^{-}\right)=0$. If not, investigate the Q -matrix using the SVD. Characterize those vectors that can be set up by an impulsive input.
(d) Find the closest vector in norm that can be set up by an impulsive input.
(e) Repeat the above problem for $\mathrm{b}_{3}=1$.

Remark: Use MATLAB as appropriate for solving this problem.
10. (4th order ccf) Repeat/investigate problem 9 using the controllable canonical form.

## II. ObSERVABILITY Problems

1. (Time-invariant case) Suppose an input $u(t)=\left[\begin{array}{ll}1 & 1\end{array}\right]^{T} 1^{+}(t)$ excites the state model

$$
\begin{aligned}
& \dot{x}(\mathrm{t})=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 2
\end{array}\right] \mathrm{x}(\mathrm{t})+\left[\begin{array}{ll}
1 & 0 \\
0 & 0 \\
0 & 1
\end{array}\right] \mathrm{u}(\mathrm{t}) \\
& \mathrm{y}(\mathrm{t})=\left[\begin{array}{lll}
1 & 0 & 1 \\
0 & 1 & 0
\end{array}\right] \mathrm{x}(\mathrm{t})
\end{aligned}
$$

(a) Suppose $x(1)=\left[\begin{array}{lll}2 e & e & e^{2}\end{array}\right]^{T}$. Compute $\mathrm{x}(\mathrm{t})$ and $\mathrm{y}(\mathrm{t})$ for $\mathrm{t} \geq 0$. Also compute $\mathrm{x}(0)$.
(b) In the presence of sensor and actuator noise and modeling errors, the response is measured as

$$
y(t)=\left[\begin{array}{c}
2.1 e^{t}+1.4 e^{2 t}-1.4 \\
0.98 e^{-t}-0.02
\end{array}\right] 1^{+}(t)
$$

Develop the necessary equations for finding $x(1)$. Given the above measurements, are the equations consistent at $\mathrm{t}=1$ ?
(c) Compute the least squares solution to the equations developed in (b). How does your answer compare with the value of $x(1)$ given in part (a).
(d) Compute the solution using any other left inverse of R and check that the resulting error, \| $\mathrm{Rx}(1)$ -$[\mathrm{Y}(1)-\mathrm{TU}(1)] \|_{2}$, is larger than the error associated with your answer in (b). Note that this norm can be computed in Matlab using the command norm(v) for the appropriate vector v .
2. (Time-varying $\mathbf{C}(\mathbf{t})$ ) Consider the system

$$
\begin{gathered}
\dot{x}(t)=A x(t)+B u(t) \\
y(t)=C(t) x(t)+D u(t)
\end{gathered}
$$

(a) Using only $\mathrm{y}(\mathrm{t})$ and $\dot{\mathrm{y}}(\mathrm{t})$ and the known input and its derivatives, derive a set of equations in $\mathrm{A}, \mathrm{B}$, $\mathrm{C}(\mathrm{t})$, and D whose solution will yield the state $\mathrm{x}(\mathrm{t})$.
(b) Suppose that

$$
A=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right] ; B=\left[\begin{array}{cc}
1 & 0 \\
0 & 0 \\
0 & 1
\end{array}\right] ; C=\left[\begin{array}{ccc}
\mathrm{e}^{\mathrm{t}} & 0 & 0 \\
0 & 1 & \mathrm{e}^{\mathrm{t}}
\end{array}\right] ; \mathrm{D}=\left[\begin{array}{cc}
0 & 0 \\
0 & 0
\end{array}\right] ; \text { and } u(\mathrm{t})=\left[\begin{array}{l}
1 \\
1
\end{array}\right] \mathrm{e}^{-\mathrm{t}}
$$

It is known that the response to a particular $\mathrm{x}(0)$ for $t \geq 0$ is

$$
y(t)=\left[\begin{array}{c}
-1 \\
e^{-t}-1
\end{array}\right]
$$

Again, using only $\mathrm{y}(\mathrm{t})$ and $\dot{\mathrm{y}}(\mathrm{t})$ compute the initial state $\mathrm{x}(0)$. Note that normally one would also form $\ddot{y}(\mathrm{t})$. However, a solution to the problem is often achievable without constructing all the derivatives of $\mathrm{y}(\mathrm{t})$ and $\mathrm{u}(\mathrm{t})$, as in this problem.
(c) Find a lower bound on the number of derivatives of $y(t)$ (including the zero-th order derivative) needed for determining a unique solution the general observability problem assuming B is $\mathrm{nxm}, \mathrm{A}$ is nxn, and $C$ is rxn.
3. (Time-varying $\mathbf{C}(\mathbf{t}))$ Consider the time-varying state model

$$
\begin{gathered}
\dot{x}(t)=A x(t)+B u(t) \\
y(t)=C(t) x(t)+D u(t)
\end{gathered}
$$

where A is 2 x 2 , B is 2 x 1 , and D is 1 x 1 while $\mathrm{C}(\mathrm{t})$ is a time-varying matrix of dimension $1 \times 2$.
(a) Suppose $y(t)$ and $u(t)$ are known for all time but no initial state vector is known. Derive a set of two linear algebraic equations whose solution at any time $t$ will yield the unknown state vector $x(t)$. Put the equations in matrix form.
(b) Find $x(0)$ under the suppositions that $u(t)=\sin (t) 1^{+}(t+1)$ and $y(t)=[t \exp (t)+1] 1^{+}(t+1)$ and that

$$
A=\left[\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right] ; B=\left[\begin{array}{l}
1 \\
2
\end{array}\right] ; C=\left[\begin{array}{ll}
t-1 & 1
\end{array}\right] ; D=[1]
$$

4. (Time-varying $\mathbf{C}(\mathbf{t}) \& \mathbf{D}(\mathbf{t}))$ A particular physical process has the linear time varying state model

$$
\begin{gathered}
\dot{x}(t)=A x(t)+B u(t) \\
y(t)=C(t) x(t)+D(t) u(t)
\end{gathered}
$$

where $x(t) \in R^{2}$ and $u(t) \in R$ for each $t$.
(a) Derive a set of equations from input-output measurements whose solution would yield $x(t)$. Put equations in matrix form.
(b) Suppose $\mathrm{u}(\mathrm{t})=\mathrm{t} 1^{+}(\mathrm{t})$ and

$$
A=\left[\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right] ; B=\left[\begin{array}{l}
0 \\
1
\end{array}\right] ; C=\left[\begin{array}{ll}
1 & t
\end{array}\right] ; D=[1-2 t]
$$

Note: D is 1 x 1 . It was found that after exhaustive measurements and checks that at $\mathrm{t}=1$ second, $y(1)=0$ and $\frac{d y(1)}{d t}=2 e-5$. Compute $x(1)$ in terms of $e$.
5. Consider the time varying state model

$$
\begin{gathered}
\dot{x}(t)=A x(t)+B u(t) \\
y(t)=C(t) x(t)+D(t) u(t)
\end{gathered}
$$

where $A$ is $3 \times 3, B$ is $3 \times 1, C(t)$ is $2 \times 3$, and $D(t)$ is $2 \times 1$.
(a) Derive a set of equations to determine $x(t)$ from input measurements and the information $y(t)$ and $\dot{y}(t)$. Put in matrix form.
(b) Find $x(1)$ when $u(t)=t 1^{+}(t), y(1)=\left[\begin{array}{ll}1 & 0\end{array}\right]^{T}$, and $\dot{y}(1)=\left[\begin{array}{ll}9 & 2\end{array}\right]^{T}$.

$$
C(t)=\left[\begin{array}{ccc}
2 t^{2}-1 & 1-2 t & 0 \\
0 & 0 & t-1
\end{array}\right], D(t)=\left[\begin{array}{c}
1-2 t \\
0
\end{array}\right], A=\left[\begin{array}{lll}
2 & 1 & 0 \\
1 & 2 & 0 \\
0 & 0 & 1
\end{array}\right], B=\left[\begin{array}{l}
2 \\
1 \\
1
\end{array}\right]
$$

(c) Discuss the solvability of the equations developed in (a) under the condition that the data is perturbed.
6. Consider the time varying state model ( A is $2 \times 2$, B is $2 \times 1$, and $\mathrm{C}(\mathrm{t})$ is $1 \times 2$ ) as specified below

$$
\begin{gathered}
\dot{x}(t)=A x(t)+B u(t) \\
y(t)=C(t) x(t)+D(t) u(t)
\end{gathered}
$$

(a) Derive a set of equations to determine $\mathrm{x}(\mathrm{t})$ from input measurements and the information $\mathrm{y}(\mathrm{t})$ and $\dot{y}(\mathrm{t})$. Put in matrix form.
(b) Suppose $\mathrm{C}(\mathrm{t})=[\sin (0.5 \pi \mathrm{t})-(4 / \pi) \cos (0.5 \pi \mathrm{t})]$, $\mathrm{D}(\mathrm{t})=[\sin (0.5 \pi \mathrm{t})]$, $\mathrm{u}(\mathrm{t})=$ $\exp (\mathrm{t}-1) 1^{+}(\mathrm{t}), \mathrm{y}(1)=\dot{y}(1)=1$, and
$\mathrm{A}=\left\lfloor\begin{array}{cc}1 & 1 \\ 1 & -1\end{array}\right\rfloor$ and $\mathrm{B}=\left\lfloor\begin{array}{l}1 \\ 1\end{array}\right\rfloor$
Find $x(1)$.
(c) Construct an impulsive input to set up $\mathrm{x}(1)$ assuming $\mathrm{x}\left(1^{-}\right)=0$.
7. Suppose an input $u(t)=\left[\begin{array}{ll}1 & 1\end{array}\right]^{T} 1^{+}(t)$ excites the state model

$$
\begin{gathered}
\dot{x}(t)=\left[\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 1 \\
-1 & 0 & -1
\end{array}\right] x(t)+\left[\begin{array}{ll}
1 & 0 \\
0 & 0 \\
0 & 1
\end{array}\right] u(t) \\
y(t)=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right] x(t)
\end{gathered}
$$

Because of limited sensor memory and noise, you could only observe that $y(1)=\left[\begin{array}{ll}1 & 2\end{array}\right]^{\mathrm{T}}$ and $\dot{y}(1)=\left[\begin{array}{ll}2 & 2\end{array}\right]^{\mathrm{T}}$.
(a) ( $\mathbf{1 0} \mathbf{~ p t s ) ~ C o n s t r u c t ~ a ~ s e t ~ o f ~ e q u a t i o n s ~ w h o s e ~ s o l u t i o n ~ w i l l ~ y i e l d ~ t h e ~ b e s t ~ e s t i m a t e ~ f o r ~} \mathrm{x}(1)$.
(b) (2 pts) Are the equations consistent/inconsistent?
(c) (10 pts) Construct the best estimate for $\mathrm{x}(1)$. Specify the Moore-Penrose pseudo left inverse of your R-matrix if it exists.
8. Consider the system

$$
\begin{gathered}
\dot{x}(t)=A x(t)+B u(t) \\
y(t)=C(t) x(t)+D u(t)
\end{gathered}
$$

(a) Using only $y(t)$ and $\dot{\mathrm{y}}(\mathrm{t})$ and the known input and its possible) derivatives, derive a set of equations in $A, B, C(t)$, and $D$ whose solution will yield the state $x(t)$.
(b) Suppose that $\mathrm{u}(\mathrm{t})=1^{+}(\mathrm{t})$,

$$
A=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right], B=\left[\begin{array}{l}
1 \\
1 \\
0
\end{array}\right], C(t)=\left[\begin{array}{ccc}
1 & -t & 0 \\
0 & 1 & -1
\end{array}\right], D=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

It is known that the response to a particular $\mathrm{x}(0)$ is

$$
y(t)=\left[\begin{array}{c}
1+t-2 t^{2} \\
4 t-1
\end{array}\right]
$$

Again, using ONLY $y(t)$ and $\dot{y}(t)$ compute the initial state $x(0)$.
(c) If because of noise $\dot{y}(0)=\left[\begin{array}{ll}1.5 & 3.5\end{array}\right]^{\mathrm{T}}$, are the equations consistent? Explain what one would solve for and how?

## Solution 8:

(a) $y(t)=C(t) x(t)+D u(t)$
$\dot{y}(t)=\dot{C}(t) x(t)+C(t) \dot{x}(t)=[\dot{C}(t)+C(t) A] x(t)+C(t) B u(t)$
In matrix form
$\left[\begin{array}{c}y(t) \\ \dot{y}(t)\end{array}\right]=\left[\begin{array}{c}C(t) \\ \dot{C}(t)+C(t) A\end{array}\right] x(t)+\left[\begin{array}{cc}D & 0 \\ C B & D\end{array}\right]\left[\begin{array}{c}u(t) \\ \dot{u}(t)\end{array}\right]$
(b) $\left[\begin{array}{c}1 \\ -1 \\ -- \\ 0 \\ 3\end{array}\right]=\left[\begin{array}{c}1 \\ -1 \\ -- \\ 1 \\ 4\end{array}\right]-\left[\begin{array}{ll}0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0\end{array}\right]\left[\begin{array}{c}1 \\ \delta(\theta)\end{array}\right]=\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{array}\right] \mathrm{x}(0)$
(c) (i) No.
(ii) One solves for $\mathrm{x}(0)$ such that $\|y(0)-\mathrm{Tu}(0)-\mathrm{Rx}(0)\|_{2}$
is minimized. This is achieved using the MP pseudo inverse $\mathrm{R}^{+}[\mathrm{y}(0)-\mathrm{Tu}(0)]=\mathrm{x}(0)$
If a left inverse exists, then $R^{+}=\left(R^{T} R\right)^{-1} R^{T}$

## III. Controllability Problems

1. (State control using different inputs) Consider the scalar differential equation $\dot{x}(\mathrm{t})=\mathrm{ax}(\mathrm{t})+\mathrm{b}$ $\mathrm{u}(\mathrm{t})$, in which $\mathrm{b} \neq 0$.
(a) find an input of the form $u(t)=\alpha 1^{+}(t)$ which will drive a given $x\left(t_{0}\right)$ to a desired $x\left(t_{1}\right)$ assuming $0<$ $\mathrm{t}_{0}<\mathrm{t}_{1} ;$
(b) if $u(t)=\alpha \delta\left(t-t_{0}\right)$, find the value of $\alpha$ which will drive a given $x\left(t_{0}\right)$ to a desired $x\left(t_{1}\right)$;
(c) if $u(t)=\alpha \exp \left(-a\left(t-t_{0}\right)\right] 1^{+}\left(t-t_{0}\right)$, find the value of a which will drive a given $x\left(t_{0}\right)$ to a desired $\mathrm{x}\left(\mathrm{t}_{1}\right)$.
2. (Time-invariant case) A state model for a particular system is

$$
\left[\begin{array}{l}
\dot{x}_{1}(\mathrm{t}) \\
\dot{\mathrm{x}}_{2}(\mathrm{t}) \\
\dot{\mathrm{x}}_{3}(\mathrm{t})
\end{array}\right]=\left[\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{array}\right]\left[\begin{array}{l}
\mathrm{x}_{1}(\mathrm{t}) \\
\mathrm{x}_{2}(\mathrm{t}) \\
\mathrm{x}_{2}(\mathrm{t})
\end{array}\right]+\left[\begin{array}{ll}
0 & \mathrm{~b} \\
0 & 0 \\
1 & 0
\end{array}\right] \mathrm{u}(\mathrm{t})
$$

It is necessary to drive $x\left(0^{-}\right)$to $x\left(0^{+}\right)=\left[\begin{array}{lll}1 & 1 & 1\end{array}\right]^{\mathrm{T}}$ by an impulsive input.
(a) What is the structure of the impulsive input?
(b) What is the equation for determining the coefficients of the impulses in part (a)? (Do NOT derive.)

Explicitly determine the Q matrix.
(c) For what values of $b$ does there NOT exist a solution. Explain or justify.
(d) If there does exist a solution, determine the minimum energy input.
(e) Determine a basis for the null space of Q , assuming there exists a solution. It should depend on Q .
(f) Characterize the set of all solutions. Why does your answer characterize the entire set of solutions?
3. (Restricted control) Consider the state dynamics

$$
\dot{x}(t)=\left[\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right] x(t)+\left[\begin{array}{ll}
1 & 0 \\
0 & 0 \\
0 & 1
\end{array}\right] u(t)
$$

(a) It is desired to set up the initial condition $x\left(0^{+}\right)=\left[\begin{array}{lll}1 & 2 & 3\end{array}\right]^{\mathrm{T}}$ given that $\mathrm{x}\left(0^{-}\right)$ $=\left[\begin{array}{lll}0 & 1 & 2\end{array}\right]^{\mathrm{T}}$ using only the impulsive input

$$
u(t)=\xi_{0} \delta(t)+\xi_{1} \dot{\delta}(t)
$$

for appropriate $\xi_{0}$ and $\xi_{1} \in \mathrm{R}^{2}$. Determine the minimum energy input of this form that will achieve the desired initial condition change by using the appropriate right inverse to achieve the answer.
(b) Characterize the set of all possible solutions to the problem.
(c) What values of the quantity $\left[\mathrm{x}(\mathrm{t})-\mathrm{x}\left(0^{-}\right)\right], \mathrm{t}>0$, are possible if the input is restricted to be of the form

$$
\mathrm{u}(\mathrm{t})=\mathrm{K}\left[\begin{array}{l}
1 \\
1
\end{array}\right] 1^{+}(\mathrm{t})
$$

where K is a user-chosen constant.
4. (Time varying $B(t)$ with restricted input ${ }^{1}$ ) Consider the state dynamics

$$
\dot{x}(t)=\left[\begin{array}{ccc}
1 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & -1
\end{array}\right] x(t)+\left[\begin{array}{cc}
t & 0 \\
0 & 1 \\
-t & 0
\end{array}\right] u(t)
$$

whose solution is given by

$$
x(t)=e^{A\left(t-t_{0}\right)} x\left(t_{0}\right)+\int_{t_{0}}^{t} e^{A(t-q)} B(q) u(q) d q
$$

(a) Characterize the set of states $x\left(0^{+}\right)$reachable from $x\left(0^{-}\right)=\left[\begin{array}{lll}0 & 0 & 0\end{array}\right]^{\mathrm{T}}$ using only the impulsive input

$$
u(t)=\xi_{0} \delta(t)
$$

for appropriate $\xi_{0} \in \mathrm{R}^{2}$.
(b) Characterize the set of states $x\left(0^{+}\right)$reachable from $x\left(0^{-}\right)=\left[\begin{array}{lll}0 & 0 & 0\end{array}\right]^{\mathrm{T}}$ using only the impulsive input

$$
u(t)=\xi_{1} \dot{\delta}(t)
$$

for appropriate $\xi_{1} \in \mathrm{R}^{2}$.
(c) Characterize the set of states $x\left(0^{+}\right)$reachable from $x\left(0^{-}\right)=\left[\begin{array}{lll}0 & 0 & 0\end{array}\right]^{\mathrm{T}}$ using the impulsive input

$$
u(t)=\xi_{0} \delta(t)+\xi_{1} \dot{\delta}(t)
$$

for appropriate $\xi_{0}$ and $\xi_{1} \in \mathrm{R}^{2}$.
(d) Determine the minimum energy input of the form given in part (c) that will drive $\mathrm{x}\left(0^{-}\right)=$
$\left[\begin{array}{lll}1 & 1 & 1\end{array}\right]^{\mathrm{T}}$ to $\mathrm{x}\left(0^{+}\right)=\left[\begin{array}{lll}3 & -2 & -1\end{array}\right]^{\mathrm{T}}$.
(e) Characterize the set of all possible solutions to the problem of part (d).
(f) Repeat the above parts when the input and initial conditions are shifted to the right by 1 unit, i.e., t--$>(\mathrm{t}-1)$ and $0^{-}$goes to $1^{-}$while $0^{+}$goes to $1^{+}$. Note that for time varying systems the ability to set up an initial condition is interval dependent whereas for a time invariant system it is not.
5. (Time-varying $\mathbf{B}(\mathbf{t})$ with restricted input ${ }^{2}$ ) Consider the state dynamics

$$
\dot{x}(t)=\left[\begin{array}{cc:c}
0 & 2 & 0 \\
0 & 0 & 0 \\
\hdashline 0 & 0 & 0
\end{array}\right] x(t)+\left[\begin{array}{cc}
0 & 0 \\
2 t & 0 \\
\hdashline 0 & 1
\end{array}\right] u(t)
$$

(a) Find $e^{A t}$.
(b) Find, if possible, an impulsive input of the form $u(t)=\xi_{0} \delta(t-1)+\xi_{1} \dot{\delta}(t-1)$ which will drive $\mathrm{x}\left(1^{-}\right)$ $=0$ to $x\left(1^{+}\right)=\left[\begin{array}{lll}1 & 1 & 1\end{array}\right]^{\mathrm{T}}$.

[^0](c) Characterize, to the extent possible, the set of all possible coefficients $\left[\xi_{0} \quad \xi_{1}\right]^{\mathrm{T}}$ that will achieve the control objective of part (b).

(d) If $x(1)=\left[\begin{array}{lll}1 & 1 & 1\end{array}\right]^{\mathrm{T}}$, find the state-response for $u(t)=\left[\begin{array}{ll}1 & 1\end{array}\right]^{\mathrm{T}} 1^{+}(t)$ given that

$$
x(t)=e^{A\left(t-t_{0}\right)} x\left(t_{0}\right)+\int_{t_{0}}^{t} e^{A(t-q)} B(q) u(q) d q
$$

6. Consider the state dynamics

$$
\dot{x}(t)=A x(t)+B(t) u(t), x\left(t_{0}\right)=x_{0}
$$

(a) What is the structure of the solution of this equation?
(b) If the input is

$$
u(t)=\xi_{0} \delta(t)+\xi_{1} \dot{\delta}(t)
$$

derive a set of equations using only this input whose solution will yield the coefficients $\xi_{0}$ and $\xi_{1}$.
(c) Suppose

$$
A=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right] \text { and } B(t)=\left[\begin{array}{cc}
1-t & 0 \\
0 & 1 \\
0 & 1-t
\end{array}\right]
$$

It is desired to set up the initial condition $\mathrm{x}\left(1^{+}\right)=\left[\begin{array}{lll}1 & 2 & 3\end{array}\right]^{\mathrm{T}}$ given that $\mathrm{x}\left(1^{-}\right)$
$=\left[\begin{array}{lll}0 & 1 & 2\end{array}\right]^{\mathrm{T}}$ using only the impulsive input

$$
u(t)=\xi_{0} \delta(t-1)+\xi_{1} \dot{\delta}(t-1)
$$

for appropriate $\xi_{0}$ and $\xi_{1}$. Construct the minimum energy input of this form that will achieve the desired initial condition change by using the appropriate right inverse to achieve the answer.
(d) Characterize the set of all possible solutions to the problem.
7. A particular physical process has the linear time varying state model

$$
\dot{x}(t)=A x(t)+B(t) u(t)
$$

where $x(t) \in R^{2}$ and $u(t) \in R$ for each $t$.
(a) Derive using the sifting property below a set of equations for driving a given $x\left(T^{-}\right)$to a desired $x\left(T^{+}\right)$using an impulsive input of the correct form. Specify the form of the input along with the dimension of the coefficients. Recall the sifting property:

$$
\int_{T^{-}}^{T^{+}} e^{A(T-q)} B(q) \delta^{(i)}(q-T) d q=(-1)^{i} e^{A T}\left[\frac{d^{i}}{d q^{i}} e^{-A q} B(q)\right]_{q=T}
$$

(b) Find a minimum energy impulsive input to drive $x\left(1^{-}\right)=\left[\begin{array}{ll}-1 & 0\end{array}\right]^{T}$ to $x\left(1^{+}\right)=\left[\begin{array}{ll}9 & 1\end{array}\right]^{T}$ when

$$
A=\left[\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right] ; B=\left[\begin{array}{cc}
0 & t(1-t) \\
1-t & 0
\end{array}\right]
$$

(c) Characterize the set of all possible solutions to part (b).

## IV. Observability and Controllability Problems

1. (Time-invariant case) A creative graduate student in psychosomatic engineering has found that their performance on a particular state variable exam satisfies the state model dynamics

$$
\begin{aligned}
& \dot{x}(t)=\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right] x(t)+\left[\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right] u(t) \\
& y(t)=\left[\begin{array}{ll}
1 & -1
\end{array}\right] x(t)+\left[\begin{array}{ll}
1 & 0
\end{array}\right] u(t)
\end{aligned}
$$

for a given induced input stimulus $u(t)$ satisfying $u(0)=\dot{u}(0)=\left[\begin{array}{ll}1 & 1\end{array}\right]^{\mathrm{T}}$ and a response $\mathrm{y}(0)=\dot{y}(0)=1$. The student wishes to identify the state of their mind at time zero as given by the state vector $x(0)$ and to raise their consciousness level to a higher state of nirvanic insight via a new NASA developed ionic impulsive electro-psychic aura regulator causing instantaneous mind-state changes.
(a) From the above data, construct the state vector $\mathrm{x}(0)$ for the system. Write down the appropriate equation and define all terms and matrices.
(b) Find the minimum energy impulsive input which will drive the initial state computed in part (a) to the "higher" state given by $x\left(0^{+}\right)=\left[\begin{array}{ll}6.5 & 3.5\end{array}\right]^{\mathrm{T}}$. Again write down the appropriate equation and define all terms. Be sure to specifically show the input.
(c) Characterize the set of all possible solutions to (b).
2. (Time-varying $\mathbf{C}(\mathbf{t})$ and $\mathbf{D}(\mathbf{t})$; complete state response) Suppose a linear system is represented by the state model

$$
\begin{gathered}
{\left[\begin{array}{l}
\dot{x}_{1}(t) \\
\dot{x}_{2}(t)
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]+\left[\begin{array}{c}
1 \\
-1
\end{array}\right] u(t)} \\
y(t)=\left[\begin{array}{ll}
1 & 1-t
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]+\left[\ln \left(\frac{e}{1+t}\right)\right] u(t)
\end{gathered}
$$

where A and B are constant and $\mathrm{C}(\mathrm{t})$ and $\mathrm{D}(\mathrm{t})$ depend on time.
(a) In terms of the symbols $\mathrm{A}, \mathrm{B}, \mathrm{C}(\mathrm{t})$, and $\mathrm{D}(\mathrm{t})$, derive a set of equations whose solution will yield $\mathrm{x}(\mathrm{t})$ in terms of $\mathrm{y}(\mathrm{t}), \mathrm{u}(\mathrm{t})$, and their derivatives. Put in matrix form.
(b) Specify your solution in part (a) in terms of the actual $\mathrm{A}, \mathrm{B}, \mathrm{C}(\mathrm{t})$, and $\mathrm{D}(\mathrm{t})$ matrices.
(c) If $u(t)=(t+1) 1^{+}(t+1)$ and the measurements $y(0)$ and $\dot{y}(0)$ are found to be 2 and 2 respectively, find $\mathrm{x}(0)$.
(d) Find an impulsive input which will set up this $x(0)$. Compute the resulting zero-input stateresponse.
(e) Using your answer to part (d) and the input and initial condition of part (c), compute the complete state-response for $\mathrm{t} \geq 0$.
3. (Time-varying $\mathbf{C}(\mathbf{t})$ and $\mathbf{D}(\mathbf{t})$ ) Consider the time varying state model (A is $2 \times 2$, $B$ is $2 \times 1$, and $C(t)$ is $1 \times 2$ ) as specified below

$$
\begin{gathered}
\dot{x}(t)=A x(t)+B u(t) \\
y(t)=C(t) x(t)+D(t) u(t)
\end{gathered}
$$

(a) Derive a set of equations to determine $\mathrm{x}(\mathrm{t})$ from input measurements and the information $\mathrm{y}(\mathrm{t})$ and $\dot{y}(\mathrm{t})$. Put in matrix form.
(b) Suppose $C(t)=[\sin (0.5 \pi t)-(4 / \pi) \cos (0.5 \pi t)], D(t)=[\sin (0.5 \pi t)], u(t)=\exp (t-1) 1^{+}(t), y(1)=$ $\dot{y}(1)=1$, and

$$
A=\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right], B=\left[\begin{array}{l}
1 \\
1
\end{array}\right]
$$

Find $x(1)$.
(c) Construct an impulsive input to set up $\mathrm{x}(1)$ assuming $\mathrm{x}\left(1^{-}\right)=0$.
4. (Time-varying $\mathbf{C}(\mathbf{t})$ ) Consider the system

$$
\begin{gathered}
\dot{x}(t)=A x(t)+B u(t) \\
y(t)=C(t) x(t)+D u(t)
\end{gathered}
$$

where

$$
A=\left[\begin{array}{ll}
-1 & 0 \\
-1 & 0
\end{array}\right], B=\left[\begin{array}{l}
1 \\
1
\end{array}\right], C=\left[\begin{array}{ll}
e^{t} & 0 \\
0 & e^{t}
\end{array}\right], D=\left[\begin{array}{l}
1 \\
0
\end{array}\right]
$$

It is known that when $u(t)=e^{-t} 1^{+}(t+1)$, the response to a particular $x(0)$ is

$$
y(t)=\left[\begin{array}{c}
1+t+e^{-t} \\
1+t
\end{array}\right] 1^{+}(t+1)
$$

(a) Using only $y(t)$ and $\dot{y}(t)$ and the known input and its derivatives, derive a set of equations in $A, B, C(t)$, and D whose solution will yield the state $x(t)$.
(b) Substitute into the equations developed in (a), and compute the initial state $x(0)$ using the appropriate left or right inverse.
(c) Suppose that Vader's Vassels were jamming your observation equipment on Cirus 3 so that instead of the initial state computed in part (b), your calculations produced the initial state $x\left(0^{-}\right)=\left[\begin{array}{ll}2 & 2\end{array}\right]^{\mathrm{T}}$. For the given system, find an impulsive input which will drive this initial state of $x\left(\begin{array}{l}0^{+}\end{array}\right)=\left[\begin{array}{ll}0 & 0\end{array}\right]^{\mathrm{T}}$ as follows:
(i) What is the form of the impulsive input?
(ii) Either derive or write by inspection the form of the equations to be solved.
(iii) Substitute the appropriate matrices and find a solution.
(d) Characterize the set of all possible solutions to part (c).
(e) From your answer to part (d), can you find (at least guess at) the minimum energy solution to (c). Justify your answer at least intuitively.
5. (Time-varying $\mathbf{C}(\mathbf{t})$ ) Consider the state model

$$
\begin{gathered}
{\left[\begin{array}{l}
\dot{x}_{1}(t) \\
\dot{x}_{2}(t)
\end{array}\right]=\left[\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]+\left[\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right] u(t)} \\
y(t)=\left[\begin{array}{ll}
t & -t
\end{array}\right]\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t)
\end{array}\right]+\left[\begin{array}{ll}
1 & 0
\end{array}\right] u(t)
\end{gathered}
$$

(a) Derive a set of equations, specific to this system, so that with exact input and output observations one can solve these equations for the state $\mathrm{x}(\mathrm{t})$ at a particular t .
(b) The input to the system for this part is $\mathrm{u}(\mathrm{t})=\left[\begin{array}{l}1 \\ 1\end{array}\right] t 1^{+}(t)$. If it is experimentally found that $\mathrm{y}(1)=3$ and $\dot{y}(1)=7$, find $x(1)$.
(c) Forget the details of parts (a) and (b). However, refer to the above state model. Suppose it is decided to set up the initial condition $x\left(1^{+}\right)=\left[\begin{array}{ll}1 & 1\end{array}\right]^{T}$ when $x\left(1^{-}\right)=\left[\begin{array}{ll}0 & 0\end{array}\right]^{\mathrm{T}}$.
(i) Set forth and justify an appropriate set of equations to accomplish this task.
(ii) Find a minimum energy impulsive input to set up this IC. Be sure to explicitly show your input.
(iii) Characterize the set of all possible solutions.
6. (Time-varying B(t)) Consider the system

$$
\begin{gathered}
\dot{x}(t)=A x(t)+B(t) u(t) \\
y(t)=C x(t)+D u(t)
\end{gathered}
$$

where

$$
A=\left[\begin{array}{cc}
1 & 1 \\
1 & -1
\end{array}\right], B(t)=\left[\begin{array}{ll}
1 & 0 \\
1 & t
\end{array}\right], C=\left[\begin{array}{ll}
1 & 1
\end{array}\right], D=\left[\begin{array}{ll}
1 & 0
\end{array}\right]
$$

(a) Using only $\mathrm{y}(\mathrm{t})$ and $\dot{y}(\mathrm{t})$ and the known input and its derivatives, derive a set of equations in $\mathrm{A}, \mathrm{B}(\mathrm{t})$, C , and D whose solution will yield the state $\mathrm{x}(\mathrm{t})$.
(b) Having used a Hackard-Pewett vector voltmeter to determine that
$[\mathrm{y}(0), \dot{y}(\mathrm{t})]=[3,2], \mathrm{u}(0)=\left[\begin{array}{ll}1 & 0\end{array}\right]^{\mathrm{T}}$, and $\dot{u}(0)=\left[\begin{array}{ll}-1 & 0\end{array}\right]^{\mathrm{T}}$, determine the initial state $\mathrm{x}(0)$ using the equations derived in part (a).
(c) You have now programmed your model onto an analog computer and desire to set up some initial condition using the impulsive input $\mathrm{u}(\mathrm{t})=\xi_{0} \delta(\mathrm{t})+\xi_{1} \dot{\delta}(\mathrm{t})$. Derive a set of equations whose solutions will yield the weights $\xi_{0}$ and $\xi_{1}$ given that

$$
x(t)=e^{A\left(t-t_{0}\right)} x\left(t_{0}\right)+\int_{t_{0}}^{t} e^{A(t-q)} B(q) u(q) d q
$$

(e) Solve the equations in part (c) for a solution assuming your answer from part (b) is $\mathrm{x}\left(0^{+}\right)$with $\mathrm{x}\left(0^{-}\right)$ $=0$.
(f) Characterize the set of all possible solutions given the particular solution calculated in part (e).
7. Consider the system

$$
\begin{gathered}
\dot{x}(t)=A x(t)+B u(t) \\
y(t)=C(t) x(t)+D u(t)
\end{gathered}
$$

where

$$
A=\left[\begin{array}{cc}
-1 & 0 \\
-1 & 0
\end{array}\right] ; B=\left[\begin{array}{cc}
1 & 1 \\
1 & 0
\end{array}\right] ; C=\left[\begin{array}{cc}
e^{t} & 0 \\
0 & e^{t}
\end{array}\right] ; D=\left[\begin{array}{cc}
t+2 & 0 \\
0 & 0
\end{array}\right]
$$

It is known that when $u(t)=e^{-t}\left[\begin{array}{l}1 \\ 0\end{array}\right] 1^{+}(t+1)$, the possibly noisy response to a particular $\mathrm{x}(0)$ is

$$
y(t)=\left[\begin{array}{c}
1+t+e^{-t} \\
1+t
\end{array}\right] 1^{+}(t+1)
$$

(a) Using only $\mathrm{y}(\mathrm{t})$ and $\dot{\mathrm{y}}(\mathrm{t})$ and the known input and its derivatives, derive a set of equations in $\mathrm{A}, \mathrm{B}$, $\mathrm{C}(\mathrm{t})$, and $\mathrm{D}(\mathrm{t})$ whose solution will yield the state $\mathrm{x}(\mathrm{t})$. Express in matrix form.
(b) Substitute into the equations developed in (a), and compute the initial state $x(0)$, at least in the least squares sense, using the appropriate left or right inverse.
(c) Suppose that Vader's Vassels were jamming your observation equipment on Cirus 3 so that instead of the initial state computed in part (b), your calculations produced the initial state $\mathrm{x}\left(0^{-}\right)=$
$\left[\begin{array}{ll}2 & 2\end{array}\right]^{\mathrm{T}}$. For the given system, find an impulsive input which will drive this initial state of $\mathrm{x}\left(0^{+}\right)=$ $\left[\begin{array}{ll}0 & 0\end{array}\right]^{\mathrm{T}}$ as follows:
(i) What is the form of the impulsive input?
(ii) Either derive or write by inspection the form of the equations to be solved.
(iii) Substitute the appropriate matrices and find the minimum energy solution.
(d) Characterize the set of all possible solutions to part (c).

## V. Theoretical Problems

1. (State Models and SISO Differential Equation Models) Given a single-input, single-output state model, develop an algorithm to construct a differential equation model in the variables y and u. Apply your algorithm to the state model determined by the matrices

$$
\mathrm{A}=\left[\begin{array}{rr}
1 & -1 \\
1 & 1
\end{array}\right] ; \mathrm{B}=\left[\begin{array}{l}
1 \\
2
\end{array}\right] ; \mathrm{C}=\left[\begin{array}{ll}
1 & 0
\end{array}\right] ; \mathrm{D}=[1]
$$

Hint:
(i) Append the equation for $\mathrm{y}^{(\mathrm{n})}$ to equation 5.4 of text.
(ii) Write this augmented equation in the form of equation 5.5, i.e., as

$$
\hat{Y}(t)=\hat{R} x(t)+\hat{T} \hat{U}(t)
$$

or equivalently as

$$
\hat{R} x(t)-I \hat{Y}(t)+\hat{T}(t) \hat{U}(t)=0
$$

(iii) Form the matrix $[\hat{R}-\hat{I} \hat{T}]$.
(iv) Upper triangularize $\hat{\mathrm{R}}$ using row operations on the matrix $[\hat{R}-\hat{I} \hat{T}]$ so that at least the bottom row of the reduced $\hat{R}$ is zero. Equivalently, one can find a vector in the left null space of $\hat{R}$ and multiply both sides of the equation developed in (ii) by this vector transposed to eliminate $x(t)$. What remains is a relationship between y and its derivatives and $u$ and its derivatives.
2. (State Models and MIMO Differential Equation Models) Let M(D) denote a matrix whose entries are polynomials in the differentiation operator $D$, i.e., $M(D)=\left[p_{i j}(D)\right]$ where $p_{i j}(D)$ is an $n$-th order polynomial in $D$ in the $i-j$ entry of $M(D)$. Suppose $y(t)$ is an r-vector and $u(t)$ an m-vector. Let $M_{1}(D)$ be rxr and $\mathrm{M}_{2}(\mathrm{D})$ be rxm. One can represent an m-input, r-output n -th order differential equation model as $M_{1}(D) y(t)=M_{2}(D) u(t)$. Consider the usual m-input r-output linear time invariant state model

$$
\begin{aligned}
& \dot{x}(t)=A x(t)+B u(t) \\
& y(t)=C x(t)+D u(t)
\end{aligned}
$$

Generalize the procedure developed in problem 1 for obtaining a differential equation model of the form $M_{1}(D) y(t)=M_{2}(D) u(t)$ from the above state model.
3. (Controllability with Time-varying $\mathbf{B}(\mathbf{t})$ ) The time-varying state dynamics

$$
\dot{x}(t)=A x(t)+B(t) u(t)
$$

has solution

$$
x(t)=e^{A\left(t-t_{0}\right)} x\left(t_{0}\right)+\int_{t_{0}}^{t} e^{A(t-q)} B(q) u(q) d q
$$

(a) Suppose $\mathrm{B}(\mathrm{t})$ is continuous and differentiable at $\mathrm{t}=0$. Using integration by parts and induction, show that for $t>T^{-}$

$$
\int_{T^{-}}^{t} e^{A(t-q)} B(q) \delta^{(i)}(q-T) d q=(-1)^{i} e^{A T}\left[\frac{d^{i}}{d q^{i}}\left(e^{-A q} B(q)\right)\right]_{q=T}
$$

(b) Suppose the above is a second order state model, i.e., two state variables. Suppose further that the input is to be impulsive of the form

$$
u(t)=\xi_{0} \delta(t)+\xi_{1} \dot{\delta}(t)
$$

Derive a set of MATRIX equations whose solution will yield the coefficient vectors $\xi_{0}$ and $\xi_{1}$.
(c) If the system matrices are given by

$$
A=\left[\begin{array}{cc}
1 & 1 \\
-1 & 1
\end{array}\right], \quad B(t)=\left[\begin{array}{cc}
\cos (t) & \sin (t) \\
-\sin (t) & \cos (t)
\end{array}\right]
$$

Determine the minimum energy input which will drive $\mathrm{x}\left(0^{-}\right)=0$ to $\mathrm{x}\left(0^{+}\right)=\left[\begin{array}{ll}1 & 1\end{array}\right]^{\mathrm{T}}$.
(d) Characterize the set of all solutions to this problem.
4. (Reachability) The time varying state dynamics

$$
\dot{x}(t)=A x(t)+B(t) u(t)
$$

has solution

$$
x(t)=e^{A\left(t-t_{0}\right)} x\left(t_{0}\right)+\int_{t_{0}}^{t} e^{A(t-q)} B(q) u(q) d q
$$

Suppose $x(0-)=0$. The objective of this problem is to investigate what values of $x(0+)$ can be achieved for different control structures.
(a) Determine the range of $\mathrm{x}\left(0^{+}\right)$when $\mathrm{u}(\mathrm{t})=\xi_{0} \delta(\mathrm{t})$ for arbitrary $\xi_{0}$ in $\mathrm{R}^{\mathrm{m}}$.
(b) Determine the range of $\mathrm{x}\left(0^{+}\right)$when $\mathrm{u}(\mathrm{t})=\xi_{1} \dot{\delta}(\mathrm{t})$ for arbitrary $\xi_{1}$ in $\mathrm{R}^{\mathrm{m}}$.
(c) Determine the range of $x\left(0^{+}\right)$when $u(t)=\xi_{0} \delta(t)+\xi_{1} \dot{\delta}(t)$ for arbitrary $\xi_{0}$ and $\xi_{1}$ in $\mathrm{R}^{\mathrm{m}}$.
(d) Determine the range of $\mathrm{x}\left(0^{+}\right)$when $\mathrm{u}(\mathrm{t})=\xi_{0} \delta(\mathrm{t})+\xi_{1} \dot{\delta}(\mathrm{t})+\xi_{2} \ddot{\delta}(\mathrm{t})$ for arbitrary $\xi_{0}, \xi_{1}, \xi_{2}$ in $\mathrm{R}^{\mathrm{m}}$.
(e) Suppose $A$ is $n x n$ and $\operatorname{rank}\left[B A B \ldots A^{p-1} B\right]=n$ for $p<n$. True-False: the input $u(t)=$ $\sum_{i=0}^{p-1} \xi_{i} \delta^{(i)}(t)$ is sufficient to drive any given $\mathrm{x}(0-)$ to any desired $\mathrm{x}(0+)$. Justify.
5. (Controllability and Controllability Gramian) Suppose $\dot{x}(t)=A x(t)+B u(t)$. Let

$$
K=\int_{0}^{t} e^{A(t-q)} B B^{T} e^{A^{T}(t-q)} d q
$$

and assume $\mathrm{K}^{-1}$ exists. If $u(q)=B^{T} e^{A^{T}(t-q)} K^{-1} z_{0}$, determine the range of possible values of $\mathrm{x}(\mathrm{t})$ for arbitrary $z_{0}$ in $R^{n}$. The matrix $K$ is called a controllability Gramian and will be discussed in detail in Chapter 20.
6. (Numbers of equations for solvability) The well known Cayley-Hamilton Theorem says: let A be an nxn matrix with characteristic polynomial, $\pi_{A}(1)=\lambda^{n}+a_{1} \lambda^{n-1}+\ldots+a_{n}$. Then $\pi_{A}(A)=A^{n}+a_{1}$ $A^{n-1}+\ldots+a_{n-1} A+a_{n} I=[0]$, the zero matrix. The theorem implies that $A^{n}$ is a linear combination of the lower powers of A .
(a) What is $\mathrm{A}^{\mathrm{n}}$ in terms of these lower powers?
(b) Use the Cayley-Hamilton Theorem to show that

$$
\operatorname{rank}\left[\begin{array}{c}
C \\
C A \\
\vdots \\
C A^{n-1}
\end{array}\right]=\operatorname{rank}\left[\begin{array}{c}
C \\
C A \\
\vdots \\
C A^{n}
\end{array}\right]
$$

(c) Consider the zero-input dynamics

$$
\begin{aligned}
& \dot{x}=A x \\
& y=C x
\end{aligned}
$$

What does the result of (b) say about the solvability of

$$
\left[\begin{array}{c}
\mathrm{y}(\mathrm{t}) \\
\dot{y}(\mathrm{t}) \\
\vdots \\
\mathrm{y}^{(\mathrm{p})}(\mathrm{t})
\end{array}\right]=\left[\begin{array}{c}
\mathrm{C} \\
\mathrm{CA} \\
\vdots \\
\mathrm{CA}^{\mathrm{p}-1}
\end{array}\right] \mathrm{x}(\mathrm{t})
$$

Explain your reasoning.
7. (Matrix powers and null space) Let $M$ be a nxn matrix. Recall that the null space of $M$ is defined as $N(M)=\left\{x \in R^{n} \mid M x=0\right\}$. Let $V_{k}=N\left(M^{k}\right)$. Show that $V_{1} \subset V_{2} \subset \ldots \subset \mathrm{~V}_{\mathrm{k}}$ for all $\mathrm{k} \leq \mathrm{n}$. Justify why $n$ can be taken as the upper limit.
8. (Rank) (a) Prove that

$$
\operatorname{rank}\left[\begin{array}{cccc}
A_{11} & A_{12} & \cdots & A_{1 N} \\
0 & A_{22} & \ddots & \vdots \\
\vdots & \ddots & \ddots & A_{N-1, N} \\
0 & \cdots & 0 & A_{N N}
\end{array}\right] \geq \sum_{i=1}^{N} \operatorname{rank}\left(A_{i i}\right)
$$

(b) Construct examples of the equality and the inequality asserted in part (a). In both cases, the upper off-diagonal blocks should have nonzero entries.

## VI SVD Related Questions

Part 1. Answer each of the questions below from the given data: from MATLAB you obtain the SVD of a particular matrix Q as:

$$
\begin{aligned}
& »[\mathrm{U}, \mathrm{~S}, \mathrm{~V}]=\operatorname{svd}(\mathrm{Q}) \\
& \mathrm{U}= \\
& 4.0825 \mathrm{e}-01
\end{aligned}-7.0711 \mathrm{e}-01 \quad 5.735 \mathrm{e}-011 .
$$

| $\mathrm{S}=$ |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $2.4495 \mathrm{e}+00$ | 0 |  | 0 |  | 0 |
| 0 | $1.4142 \mathrm{e}+00$ |  | 0 |  | 0 |
| 0 | 0 | 0 |  | 0 |  |


| $\mathrm{V}=$ |  |  |  |
| :---: | :---: | :---: | :---: |
| $5.0000 \mathrm{e}-01$ | -5.0000e-01 | 6.7238 e | -2.1886e-01 |
| $5.0000 \mathrm{e}-01$ | $5.0000 \mathrm{e}-01$ | 2.1886e-01 | 6.7238e-01 |
| $5.0000 \mathrm{e}-01$ | -5.0000e-01 | -6.7238e-0 | $2.1886 \mathrm{e}-01$ |
| $5.0000 \mathrm{e}-01$ | $5.0000 \mathrm{e}-0$ | -2.1886e- | $-6.7238 \mathrm{e}-01$ |

## True-False Questions:

(i) Question 1: $Q$ has a left inverse. $\qquad$
(ii) Question 2: $Q$ has a right inverse. $\qquad$
(iii) Question 3: $Q$ has a pseudo inverse. $\qquad$
(iv) Question 4: If $\mathrm{Q} x=b$ is an equation which is inconsistent, then a least squares solution is given by $\mathrm{x}=\operatorname{pinv}(\mathrm{Q})^{*} \mathrm{~b}$. $\qquad$
(v) Question 5: The matrices $U$ and $V$ satisfy, $U^{*} U^{T}=I$ and $V^{T} * V=I$. $\qquad$

## Fill-in-the-blank

(vi) Question 6: The spectral norm of $Q$ is: $\qquad$
(vii) Question 7: A basis for the column space of Q is:
(viii) Question 8: A basis for the null space of Q is:

Part 2: Answer each of the questions below from the given data: from MATLAB you obtain the SVD of a particular matrix M as:

```
»[U,S,V] = svd(M)
U =
    0.5000 -0.5000 -0.6304 -0.3203
    0.5000 0.5000
    0.5000 -0.5000 0.6304 0.3203
    0.5000 0.5000 -0.3203 0.6304
S =
    2.4495 0 0
            0}1.4142 
            0}000.000
            0 0
V =
    0.4082 -0.7071 0.5774
    0.4082
    0.8165 -0.0000 -0.5774
```


## True-False ( $\mathbf{2}$ pts each, -1 for wrong answer, 0 for no answer):

(i) Question 1: M has a left inverse. $\qquad$
(ii) Question 2: M has a right inverse. $\qquad$
(iii) Question 3: $M$ has a pseudo inverse. $\qquad$
(iv) Question 4: If $\mathrm{Mx}=\mathrm{b}$ is an equation which is inconsistent, then a least squares solution is given by $x=\operatorname{pinv}(\mathrm{M}) * b$. $\qquad$ where pinv means Moore-Penrose pseudo inverse.
(v) Question 5: The matrices $U$ and $V$ satisfy, $U^{*} U^{T}=I$ and $V^{T} * V=I$. $\qquad$
Fill-in-the-blank ( 2 pts each, no penalty):
(vi) Question 6: The spectral norm of $M$ is: $\qquad$
(vii) Question 7: A basis for the column space of $M$ is:
(viii) Question 8: A basis for the null space of M is:

## 2. True-False Questions.

(i) The Moore-Penrose right inverse exists for $Q=\left[\begin{array}{llll}1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0\end{array}\right]$.
(ii) The Moore-Penrose left inverse exists for $R=\left[\begin{array}{cc}1 & 1 \\ 0 & 0 \\ 1 & -1 \\ -1 & 1\end{array}\right] \cdot \square$
(iii) Suppose the equation, $Q \xi=x$, is consistent where $Q \neq 0$ is $\mathrm{n} \times(\mathrm{rn}), \mathrm{r}, \mathrm{n} \geq 1$. A minimum norm value of $\xi$ is obtained by using the Moore-Penrose pseudo inverse of $Q$, denoted $Q^{+}$, via $\xi=Q^{+} x$. $\qquad$
(iv) Again consider the equation, $Q \xi=x$, where $Q \neq 0$ is $\mathrm{n} \times(\mathrm{rn}), \mathrm{r}, \mathrm{n} \geq 1$. If $\operatorname{rank}[Q, x]=\operatorname{rank}[Q]$, then the equations are consistent. $\qquad$
(v) If $\operatorname{rank}(\mathrm{Q})=\mathrm{n}$ where $\mathrm{Q} \neq 0$ is $\mathrm{n} \times(\mathrm{rn}), \mathrm{r}, \mathrm{n}>1$, then there exist precisely $\mathrm{n}(\mathrm{r}-1)$ independent vectors, $\mathrm{v}_{\mathrm{i}}$, such that $\mathrm{Qv}_{\mathrm{i}}=0$. $\qquad$
(vi) Let $\sigma_{1} \geq \sigma_{2} \geq \ldots \geq \sigma_{\mathrm{n}}$ be the n nonzero singular values of $\mathrm{Q} \neq 0$ is $\mathrm{n} \times(\mathrm{rn}), \mathrm{r}, \mathrm{n} \geq 1$. The longest nonzero stretch that a vector $\xi$ undergoes is $\sigma_{1}$ and the shortest nonzero stretch that $\xi$ undergoes is $\sigma_{n}$. $\qquad$
(vii) A single-input single-output linear time-invariant lumped system is in the observable canonical state model form. The associated observability matrix R is always invertible. $\qquad$
(viii) If P and Q are nonsingular matrices, then if PAQ is well defined it follows that $\operatorname{rank}(\mathrm{PAQ})$ $=\operatorname{rank}(\mathrm{A})$.
(ix) $\operatorname{rank}(A)=\operatorname{rank}\left(\mathrm{A}^{\mathrm{T}}\right)$. $\qquad$
(x) $\operatorname{rank}\left[\right.$ block-diag $\left.\left[\mathrm{A}_{11}, \mathrm{~A}_{22}, \mathrm{~A}_{33}, \ldots, \mathrm{~A}_{\mathrm{nn}}\right]\right]$. $\qquad$

$$
\text { (xi) } \operatorname{rank}\left[\begin{array}{ccccc}
A_{11} & A_{12} & \cdots & A_{1, n-1} & A_{1 n} \\
0 & A_{22} & \cdots & A_{2, n-1} & A_{2 n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & A_{n n}
\end{array}\right] \geq \sum_{i=1}^{n} \operatorname{rank}\left[A_{i i}\right]
$$

## Part 3: More True-False Questions (with Answers)

1. The Moore-Penrose right inverse exists for $Q=\left[\begin{array}{cccc}1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0\end{array}\right]$. $\qquad$
2. The Moore-Penrose pseudo left inverse exists for $R=\left[\begin{array}{ll}1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2\end{array}\right] \cdot \square$
3. Suppose the equation, $R x=b$, is inconsistent where $R \neq 0$ is $(r n) \times n, r, n \geq 1$. A least squares solution is always obtained by using the Moore-Penrose pseudo inverse of R . $\qquad$
4. Again consider the equation, $R x=b$, where $R \neq 0$ is $(r n) \times n, r, n \geq 1$. If in MATLAB, $\operatorname{rank}[R, b]=$ $\operatorname{rank}[R]$, then the equations are consistent. $\qquad$
5. A least squares solution of the equation, $\mathrm{Rx}=\mathrm{b}$, minimizes $\|\mathrm{b}-\mathrm{Rx}\|_{2}$. $\qquad$
6. A single-input single-output linear time-invariant lumped system is in the controllable canonical state model form. The associated matrix $Q=\left[\begin{array}{lll}B & A B & A^{2} B\end{array} \ldots A^{n-1} B\right]$ is always invertible. $\qquad$
7. The equation, $\mathrm{Q} \xi=\mathrm{x}$, is almost consistent if there exists a vector v such that (i) $\|\mathrm{v}\|_{2}>\|x\|_{2}$ and (ii) $(v+x)$ is in the column space of $Q$. $\qquad$
8. If two linear state models are algebraically equivalent, they are also zero-state equivalent.

ANSWERS: 1. True, 2. False, 3. True, 4. True, 5. True, 6. True, 7. False, 8. True.


[^0]:    ${ }^{1}$ See Theoretical Problem 3.
    ${ }^{2}$ See Theoretical Problem 3.

