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(15 pts) **1.** Using the definition of the Fourier transform (*not* the table of Fourier transform pairs), compute the Fourier transform of the DT signal:

$$x[n] = \left(\frac{1}{2j}\right)^{|n|}.$$

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2j}\right)^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2j}\right)^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} \left(\frac{1}{2j}\right)^{-n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2j} e^{-j\omega}\right)^{n} + \sum_{n=-\infty}^{-1} \left(\left(\frac{1}{2j}\right)^{-1} e^{-j\omega}\right)^{n}$$

NO.

$$= \frac{1}{1-\frac{1}{2j}e^{-j\omega n}} + \sum_{n=0}^{\infty} \left(\left(\frac{1}{2j}\right)^{-1} e^{-j\omega}\right)^{n-1}$$

NO.

$$= \frac{1}{1-\frac{1}{2j}e^{-j\omega n}} + \left(\left(\frac{1}{2j}\right)^{-1} e^{-j\omega}\right) \sum_{n=0}^{\infty} \left(\left(\frac{1}{2j}\right)^{-1} e^{-j\omega}\right)^{n}$$

$$= \boxed{\frac{1}{1-\frac{1}{2j}e^{-j\omega n}} + \left(\left(\frac{1}{2j}\right)^{-1} e^{-j\omega}\right)\left(\frac{1}{1-\left(\frac{1}{2j}\right)^{-1} e^{-j\omega n}}\right)}$$

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(20 pts) **2.** The Frequency response of a continuous-time LTI system is

$$H(j\omega) = \mathcal{H}(\omega) = \frac{1}{j\omega + 2}.$$

Use the convolution property of the Fourier transform to determine the response $y(t)$ when the input is $x(t) = e^{-|t|}$.

$y(j\omega) = X(j\omega)\,H(j\omega)$

$x(\tau) = e^{-|\tau|}$

$X(j\omega) = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$

$= \int_{-\infty}^{\infty} e^{-|\tau|} e^{-j\omega\tau}\, d\tau$

$= \int_{0}^{\infty} e^{-\tau} e^{-j\omega\tau}\, d\tau + \int_{-\infty}^{0} e^{\tau} e^{-j\omega\tau}\, d\tau$

$= \int_{0}^{\infty} e^{-\tau - j\omega\tau}\, d\tau + \int_{-\infty}^{0} e^{\tau - j\omega\tau}\, d\tau$

$= \int_{0}^{\infty} e^{\tau(-1-j\omega)}\, d\tau + \int_{-\infty}^{0} e^{\tau(1-j\omega)}\, d\tau$

$= \left.\frac{e^{\tau(-1-j\omega)}}{(-1-j\omega)}\right|_{0}^{\infty} + \left.\frac{e^{\tau(1-j\omega)}}{1-j\omega}\right|_{-\infty}^{0}$

$= \left(\frac{e^{\infty(-1-j\omega)}}{(-1-j\omega)} - \frac{1}{(-1-j\omega)}\right) + \left(\frac{1}{1-j\omega} - \frac{e^{\infty(1-j\omega)}}{1-j\omega}\right)$

$= -\frac{1}{(-1-j\omega)} + \frac{1}{1-j\omega}$ ✓

use table instead

$X(j\omega)H(j\omega) = x(\tau) * h(\tau) = y(t)$ NO.

$= \int_{-\infty}^{\infty} (-e^{-\tau} + e^{\tau})u(\tau)\, e^{-2(t-\tau)} u(t-\tau)\, d\tau$

$= \int_{0}^{t} e^{-\tau} e^{-2(t-\tau)} + e^{\tau} e^{-2(t-\tau)}\, d\tau$

$= e^{-2t} \int_{0}^{t} e^{-3\tau} + e^{-\tau}\, d\tau$

$= e^{-2t}\left(-\frac{e^{-3\tau}}{3} - e^{-\tau}\right)\Big|_{0}^{t}$

$= e^{-2t}\left(\left(-\frac{e^{-3t}}{3} + \frac{1}{3}\right) - (e^{-t} - 1)\right)$

$\boxed{y(t) = -\frac{e^{-5t}}{3} + \frac{e^{-2t}}{3} - e^{-3t} - e^{-2t}}$

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$x(\tau) = -e^{-\tau}u(\tau) + e^{\tau}u(\tau)$

$h(\tau) = e^{-2\tau}u(\tau)$

You already know $x(t)$. No need to invert $X(\omega)$.

$-1(1-j\omega)$

$j\omega-1 \quad j\omega+1 \quad (j\omega-1$

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(15 pts) **3.** True/False? The Fourier transform of a DT signal $x[n]$ is a periodic function, no matter what $x[n]$ is. (Justify your answer.)

True.

All FT of DT signals are periodic with period of 2π

ex FT of $a^n u[n] = \frac{1}{1-ae^{-j\omega}}$ which is periodic with period 2π

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(10 pts) **4.** A continuous-time LTI system has frequency response

$$H(j\omega) = \frac{j\omega + 4}{(j\omega + 2)(j\omega + 3)}.$$

Derive a differential equation representing this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

$H(j\omega) X(j\omega) = Y(j\omega) \quad \Rightarrow \frac{s+4}{(s+2)(s+3)} = \frac{s+4}{s^2+5s+6}$

$\frac{d}{dt} x(t) = j\omega X(\omega) \quad (16)$

$\frac{d^2 y(t)}{dt^2} + 5 \frac{d\, y(t)}{dt} + 6 y(t) = \frac{d\, x(t)}{dt} + 4 x(t) \Rightarrow$ using (16)

$\Rightarrow j\omega^2 Y(\omega) + 5 j\omega Y(\omega) + 6 Y(\omega) = j\omega^2 X(\omega) + 4 X(\omega)$

$= Y(j\omega)(j\omega^2 + 5j\omega + 6) = X(j\omega)(j\omega + 4)$

$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \Rightarrow \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 4}{j\omega^2 + 5j\omega + 6}$

$= \frac{j\omega + 4}{(j\omega+2)(j\omega+3)}$

SO This is the differential eq

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(10 pts) **5.** A CT signal $x(t)$ has Fourier transform

$$\mathcal{X}(\omega) = -2e^{(j-1)\omega}u(\omega+1).$$

Denote by $y(t)$ the signal obtained by delaying $x(t)$ by six seconds. Sketch a graph representing the magnitude $|\mathcal{Y}(\omega)|$ of the Fourier transform $\mathcal{Y}(\omega)$ of $y(t)$. (Justify your answer.)

$y(t) = x(t-6)$ when $x(t-t_0)$ happens $X(\omega) \to e^{-6j\omega}X(\omega)$

$y(\omega) = e^{-6j\omega}\left(-2e^{(j-1)\omega}u(\omega+1)\right)$

$= e^{-6j\omega}(-2e^{j\omega}e^{-\omega}u(\omega+1)$

$= -2e^{-5j\omega-\omega}u(\omega+1)$

$= -2e^{-5j\omega}e^{-\omega}u(\omega+1)$

???

$|y(\omega)| = |-2e^{-\omega}| = \sqrt{(-2e^{-\omega}\cdot -2e^{-\omega}) + (-2e^{-\omega}\cdot -2e^{-\omega})}$

$u(\omega+1)$ $= \sqrt{4e^{-2\omega} + 4e^{-2\omega}}$

$= \sqrt{8}\,e^{-2\omega}$