Convolution Property

$$y(t) = S\{x(t)\} = x(t) * h(t)$$

$$x(t) = e^{j\omega t} \Rightarrow y(t) = H(\omega)e^{j\omega t}$$

Look at the synthesis for $x(t)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$y(t) = S\{x(t)\} = S\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega\right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,S\{e^{j\omega t}\}\,d\omega$$

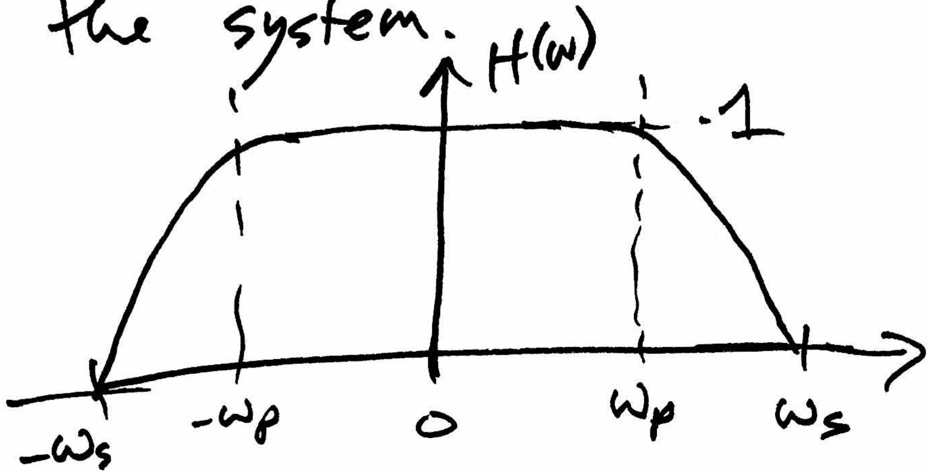
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(\omega)H(\omega)}_{Y(\omega)} e^{j\omega t}\,d\omega$$

$$x(t) * h(t) \overset{CTFT}{\longleftrightarrow} X(\omega)H(\omega)$$

Convolution in time is multiplication in frequency.

$H(\omega) = \mathcal{F}\{h(t)\}$

Just $h(t)$, $H(\omega)$ completely characterizes the system.



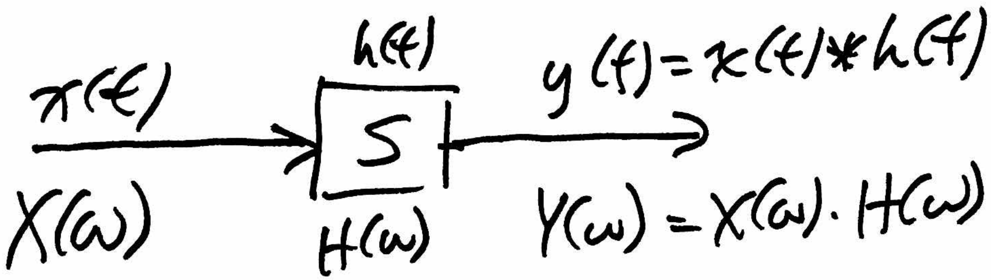
$H(\omega)$ ~ frequency response

How the system responds to particular frequencies.

$|\omega| < \omega_p$ ~ passband

$|\omega| > \omega_s$ ~ stopband

$\omega_p \to \omega_s$ ~ transition band

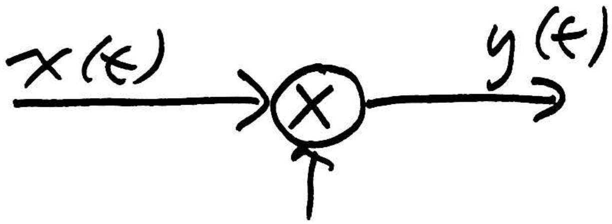


Multiplication property

$$x(t)\cdot y(t) \xleftrightarrow{\text{CTFT}} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta)\, Y(\theta-\omega)\, d\theta = \frac{1}{2\pi} X(\omega) * Y(\omega)$$

Multiplication in time is convolution in frequency.

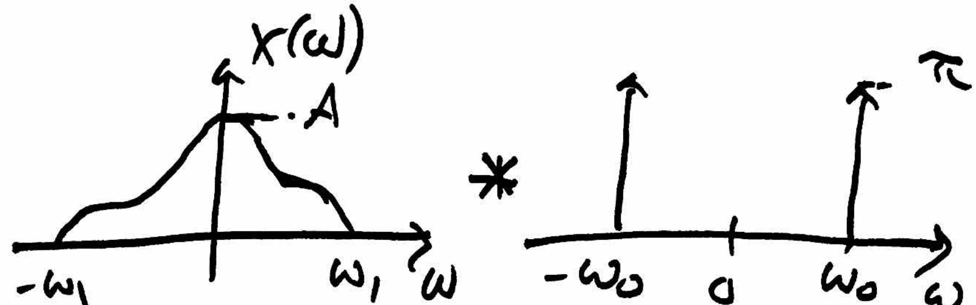
<u>Ex</u>



Modulator

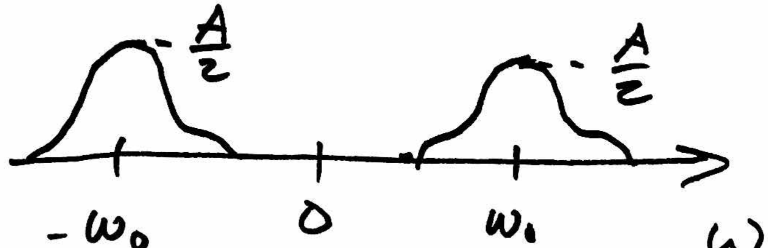
$c(t) = \cos\omega_0 t$

$y(t) = x(t)\cdot c(t) \longleftrightarrow$



Convolving with a delta scales and shifts a function.

$Y(\omega) =$



$$Y(\omega) = \frac{1}{2\pi} X(\omega) * \underset{C(\omega)}{H(\omega)}$$

$$= \frac{1}{2\pi}\left(X(\omega) * \left(\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\right)\right)$$

$$= \frac{1}{2}\left(X(\omega) * \delta(\omega-\omega_0) + X(\omega) * \delta(\omega+\omega_0)\right)$$

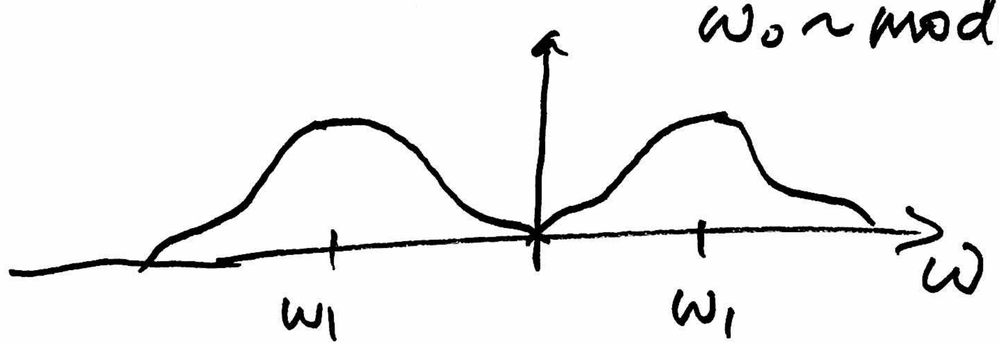
$$= \frac{1}{2}\left(X(\omega-\omega_0) + X(\omega+\omega_0)\right)$$

★ $$x(t)\cos\omega_0 t \xleftrightarrow{CTFT} \frac{1}{2}\left(X(\omega-\omega_0) + X(\omega+\omega_0)\right)$$

what if $\omega_0 = \omega_1$?

ω_1 ~ highest nonzero freq. of $X(\omega)$

ω_0 ~ modulating frequency



$\omega_0 > \omega_1$ to prevent overlap around $\omega = 0$

<u>Ex</u> $\text{rep}_T\{x(t)\} \xleftrightarrow{CTFT} ?$

$$\text{rep}_T\{x(t)\} = \sum_{k=-\infty}^{\infty} x(t-kT) = \sum_{k=-\infty}^{\infty} x(t) * \delta(t-kT)$$

$$= x(t) * \underbrace{\left(\sum_{k=-\infty}^{\infty} \delta(t-kT)\right)}_{p_T(t)\ \sim\ \text{delta/impulse train}}$$

Need $P_T(\omega)$, but $p_T(t)$ is periodic so ~~we~~ we need to ~~the~~ use the Fourier series.

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\,dt \quad \leftarrow \text{plug in } p_T(t)$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)\cdot 1\,dt$$

$$= \frac{1}{T}$$

$$p_T(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\frac{2\pi}{T}t}$$

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$$

$$P_T(\omega) = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\frac{2\pi}{T}t}\right\}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} \mathcal{F}\left\{e^{jk\frac{2\pi}{T}t}\right\}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} 2\pi\,\delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$Y(\omega) = \mathcal{F}\{\mathrm{rep}_T\{x(t)\}\} = \frac{2\pi}{T} X(\omega)\cdot\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{T}\right)$$

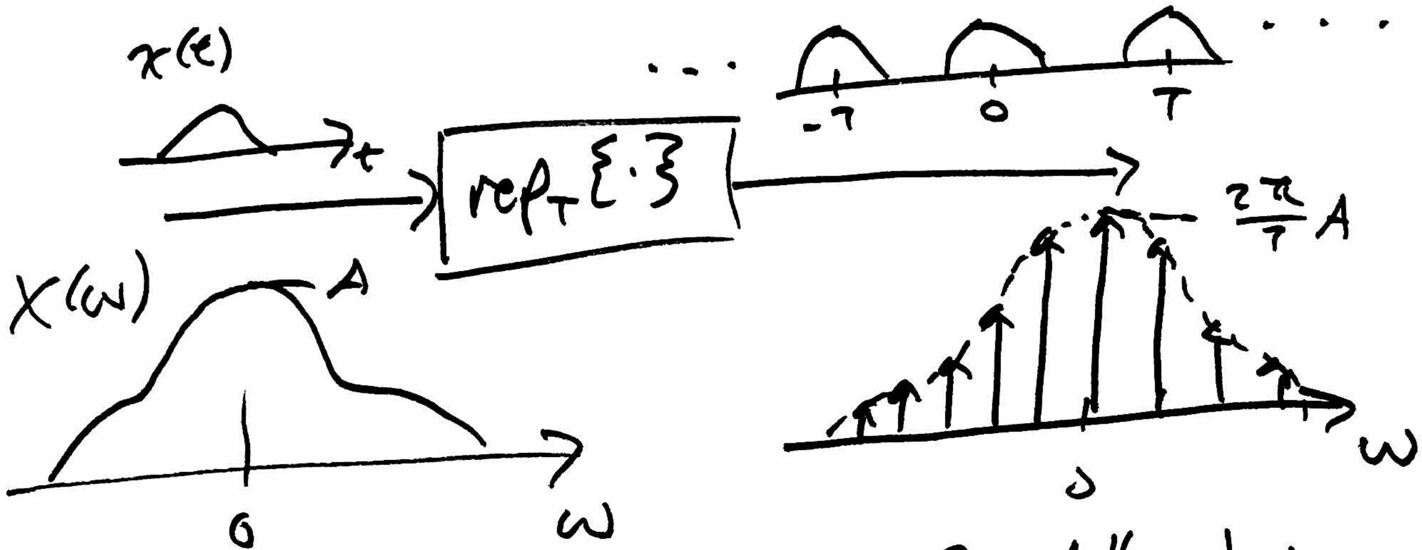
$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} X(\omega)\,\delta\left(\omega - k\frac{2\pi}{T}\right) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} X\left(k\frac{2\pi}{T}\right)\delta\left(\omega - k\frac{2\pi}{T}\right)$$

Purdue notation:

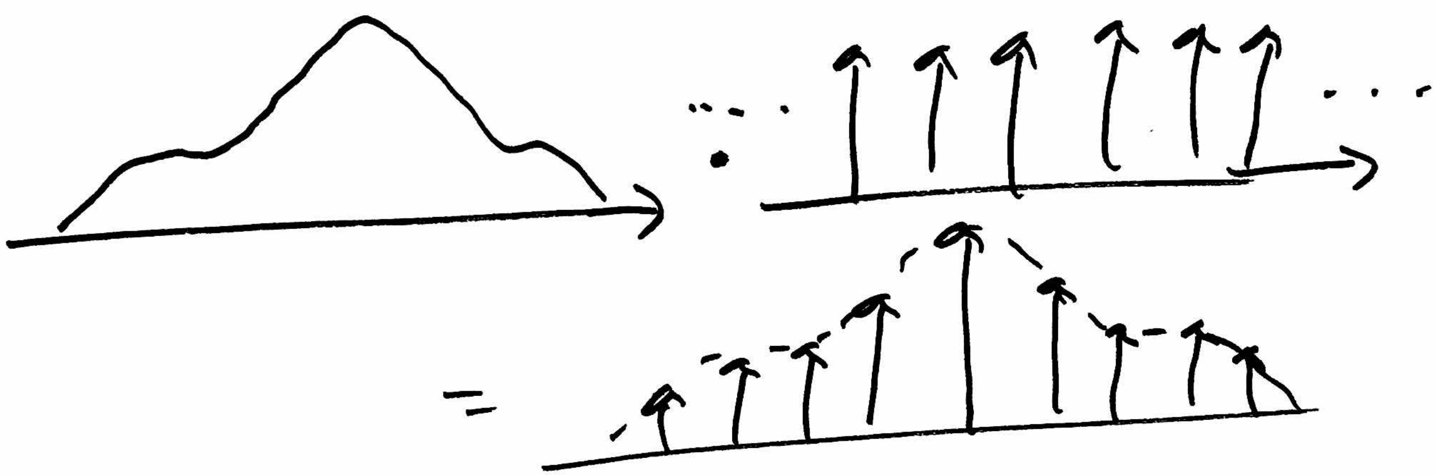
$$\mathrm{rep}_T\{x(t)\} \xleftrightarrow{\text{CTFT}} \frac{2\pi}{T}\,\underline{\mathrm{comb}_{\frac{2\pi}{T}}}\{X(\omega)\}$$

convolve with delta train, spacing of T

multiply with a delta train, spacing $\frac{2\pi}{T}$



$X(\omega)$ is an envelope for delta train



Ex $\text{comb}_T\{x(t)\} \xleftrightarrow{\text{CTFT}} ?$

$$\text{comb}_T\{x(t)\} \longleftrightarrow \frac{1}{T}\,\text{rep}_{\frac{2\pi}{T}}\{X(\omega)\}$$