Ex Let X be an exponential r.v. with mean $1/\lambda$. Find $\varphi_X(\omega)$

$$\varphi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

$$= \int_{-\infty}^{\infty} \lambda e^{-\lambda x} u(x)\, e^{j\omega x}\, dx$$

$$= \int_{0}^{\infty} \lambda e^{-(\lambda - j\omega)x}\, dx$$

$$= \frac{\lambda}{\lambda - j\omega}$$

Uniqueness Theorem of Characteristic Functions:

Let X_1, X_2 be r.v.s with pdf $f_{X_1}(x)$ and $f_{X_2}(x)$, and characteristic functions $\varphi_{X_1}(\omega)$ and $\varphi_{X_2}(\omega)$. Then

$$\varphi_{X_1}(\omega) = \varphi_{X_2}(\omega) \overset{\text{(equivalent)}}{\Longleftrightarrow} f_{X_1}(x) = f_{X_2}(x)$$

Ex The characteristic function of a Gaussian r.v. X with mean μ_X and variance σ_X^2

$$\varphi_X(w) = \exp(j\mu_X w - \sigma_X^2 w^2/2)$$

Find the characteristic function of $Y = aX + b$ (a, b are constants)

$$\varphi_Y(w) = E[e^{jwY}]$$

$$= E[e^{jw(aX+b)}]$$

$$= e^{jwb} E[e^{jwaX}]$$

$$= e^{jwb} \cdot \varphi_X(aw)$$

$$= e^{jwb} \exp(j\mu_X aw - \sigma_X^2 a^2 w^2/2)$$

$$= \exp(j(a\mu_X + b)w - a^2\sigma_X^2 w^2/2)$$

$\Rightarrow$ Y is Gaussian with mean $a\mu_X + b$ and variance $a^2\sigma_X^2$.

Since $f_X(x)$ and $\varphi_X(w)$ form a transform pair we would expect $\varphi_X(w)$ to contain the same information as $f_X(w)$. The moment theorem gives us a way to find the moments of X from $\varphi_X(w)$.

Moment Theorem: Let X be a r.v. with characteristic function $\varphi_X(w)$. Then the n^{th} moment of X is given by:

$$E[X^n] = \frac{1}{j^n} \frac{d^n}{dw^n} \varphi_X(w) \Big|_{w=0}$$

EX Use the moment theorem to find the 2^{nd} central moment of a Gaussian r.v. X.

Want to find $E[(X-\mu_X)^2]$

Let $Y = X - \mu_X$

$\Rightarrow$ Y is ~~ga~~ Gaussian with mean 0 and variance σ_X^2.

$$E[(X-\mu_X)^2] = E[Y^2]$$

$$= \frac{1}{j^2} \frac{d^2}{dw^2} \varphi_Y(w) \Big|_{w=0}$$

$$= \frac{1}{j^2} \frac{d^2}{dw^2} \exp(-\sigma_X^2 w^2/2) \Big|_{w=0}$$

$$= -1 \cdot \sigma_X^2 \exp(-\sigma_X^2 \omega^2/2)(\sigma_X^2 \omega^2 - 1) \Big|_{\omega=0}$$

$$= \sigma_X^2$$

Can use same approach to find $E[(X-\mu_X)^n]$, $n \geq 1$

Two Random Variables

Many random experiments involve multiple quantities of interest or ~~quantites~~ quantities that are repeatedly measured.

Ex
- Measure temperature and pressure of outdoors

  X = temperature in C°

  Y = pressure in atm

- Throw dart at dartboard (bullseye is origin)

  X = x-coordinate in m

  Y = y-coordinate in m

  R = distance to dart = $\sqrt{X^2 + Y^2}$

- Tossing two dice

  (X, Y) be the values rolled

So far, we have discussed how to calculate probabilities of events involving a single, isolated r.v.

We now discuss tools to analyze the joint behavior of two r.v.s