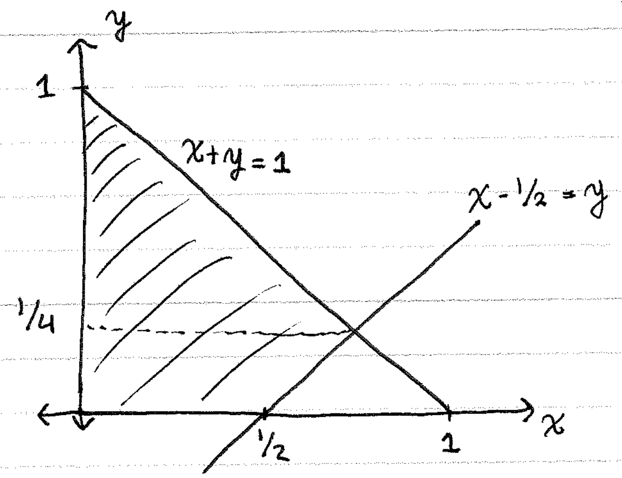
c) Find $\Pr(X - 1/2 \le Y)$

$$\Pr(X - 1/2 \le Y) = \iint_{x - 1/2 \le y} f_{X,Y}(x, y)\, dx\, dy$$



$$\Pr(X - 1/2 \le Y) = \int_0^{1/4} \int_0^{y+1/2} 6x\, dx\, dy$$

$$+ \int_{1/4}^{1} \int_0^{1-y} 6x\, dx\, dy$$

$$= \int_0^{1/4} \frac{6x^2}{2} \Big|_0^{y+1/2} dy$$

$$+ \int_{1/4}^{1} \frac{6x^2}{2} \Big|_0^{1-y} dy$$

$$= \int_0^{1/4} 3(y+1/2)^2\,dy + \int_{1/4}^{1} 3(1-y)^2\,dy$$

$$= \int_{1/2}^{3/4} 3u^2\,du + \int_0^{3/4} 3v^2\,dv$$

$$u^2 = y + 1/2 \qquad V = 1 - y$$

$$= \frac{3u^3}{3}\Big|_{1/2}^{3/4} + \frac{3v^3}{3}\Big|_0^{3/4}$$

<u>Note</u>: Let $Z = X - Y$. How can we find the cdf of Z?

$$F_Z(z) = \Pr(Z \le z)$$

$$= \Pr(X - Y \le z)$$

$$= \Pr(X - z \le Y)$$

Can repeat the analysys from c) to find $F_Z(z)$

<u>Conditional cdf, pdf, and pmf</u>

As stated before, can find the conditional pdf of X given Y as

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad f_Y(y) > 0$$

Can also derive a total probability ~~expr~~ expression for this conditional pdf

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)\, f_Y(y)$$

$$\Rightarrow \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_{-\infty}^{\infty} f_{X|Y}(x|y)\, f_Y(y)\, dy$$

$$\Rightarrow \quad f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)\, f_Y(y)\, dy$$

Total Propability for pdfs

Bayes Rule for pdfs

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)}$$

Let X and Y be discrete r.v.s. The <u>conditional pmf</u> of X given Y is

$$p_{X|Y}(x_i|y_j) = \Pr(X = x_i \mid Y = y_j)$$

$$\Rightarrow p_{X|Y}(x_i|y_j) = \frac{p_{X,Y}(x_i, y_j)}{p_Y(y_j)} \qquad (p_Y(y_j) > 0)$$

$$p_{Y|X}(y_j|x_i) = \frac{p_{X,Y}(x_i, y_j)}{p_X(x_i)} \qquad (p_X(x_i) > 0)$$

Total Propability for pmfs

$$P_Y(y_j) = \sum_{x_i} P_{Y|X}(y_j|x_i)\, P_X(x_i)$$

Bayes Rule for pmfs

$$P_{X|Y}(x_i|y_j) = \frac{P_{Y|X}(y_j|x_i)\, P_X(x_i)}{P_Y(y_j)}$$

<u>Independence of R.V.</u>

Let X and Y be r.v.s. We say X and Y are independent if

$$Pr(X \in A, Y \in B) = Pr(X \in A)\, Pr(Y \in B)$$

for all $A, B \subset \mathbb{R}$

Following statements are <u>equivalent</u>:

(1) X and Y are independent r.v.s

(2) $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$ for all x and y

(3) $f_{X|Y}(x|y) = f_X(x)$ for all x, y with $f_Y(y) > 0$

(4) $f_{Y|X}(y|x) = f_Y(y)$ for all x, y with $f_X(x) > 0$

(5) $f_{X,Y}(x,y) = a(x)\, b(y)$ for all x, y,

where $a(x)$ and $b(y)$ are functions.

If X, Y are independent then $g(X)$, $h(Y)$ are also independent. This implies

$$E[g(X)h(Y)] = E[g(X)]\,E[h(Y)]$$

$$\Rightarrow E[XY] = E[X]\,E[Y]$$

Correlation of R.V.s

Let X, Y be r.v.s. The correlation between X and Y is $E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy$

The covariance between X and Y is

$$Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y)\, f_{X,Y}(x,y)\,dx\,dy$$

Note: $Cov[X,X] = Var[X]$

$Cov[X,Y] = Cov[Y,X]$

X and Y are uncorrelated r.v.s if

$$Cov[X,Y] = 0 \iff E[XY] = E[X]\,E[Y]$$

Follows from $Cov[X,Y] = E[XY] - E[X]\,E[Y]$

Independence $\Rightarrow$ Uncorrelated

Uncorrelated $\not\Rightarrow$ Independence

The r.v.s X and Y are <u>orthogonal</u> if

$$E[XY] = 0$$

The covariance of X and Y measures the correlation between the behavior of X and Y.

$$Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$$

$$\left.\begin{array}{l} \text{If there is a tendency for} \\ (X-\mu_X) > 0 \quad \text{when} \quad (Y-\mu_Y) > 0 \\ \text{or} \\ (X-\mu_X) < 0 \quad \text{when} \quad (Y-\mu_Y) < 0 \end{array}\right\} \Rightarrow Cov[X,Y] > 0$$

$$\left.\begin{array}{l} \text{If there is a tendency for} \\ (X-\mu_X) < 0 \quad \text{when} \quad (Y-\mu_Y) > 0 \\ \text{or} \\ (X-\mu_X) > 0 \quad \text{when} \quad (Y-\mu_Y) < 0 \end{array}\right\} \Rightarrow Cov[X,Y] < 0$$