10/21/09

ECE 495N, Fall'09 GRIS 280, MWF 1130A – 1220P

HW#6: Due Friday Oct.30 in class.

Assuming ***periodic boundary conditions***, find

the density of states $D(E) = \sum_{\vec{k}} \delta(E - \varepsilon(\vec{k}))$

and the mode density $M(E) = \sum_{\vec{k}} \delta(E - \varepsilon(\vec{k})) \dfrac{\pi\hbar \left| v_x(\vec{k}) \right|}{L}$

for a two-dimensional conductor with an $\varepsilon(\vec{k})$ relation of the form

$$\varepsilon(\vec{k}) = Ak^n, \text{ where } k^2 = k_x^{\ 2} + k_y^{\ 2}$$

n being an integer, assuming

(1) that both L and W large enough that the summations over k_x and k_y can both be replaced with appropriate integrals.

(2) that L is large enough that the summation over k_x can be replaced with an appropriate integral, but W is NOT large enough to do the same for k_y.

HW 6 – Solution

1) $\mathcal{E}(K) = A K^n$

* Calculating density of state:

$$D(E) = \sum_{\vec{K}} \delta(E - \mathcal{E}(K)) = \frac{L.W}{(2\pi)^2} \int_0^{\infty} dk\, K \int_0^{2\pi} d\varphi\, \delta(E - \mathcal{E}(K))$$

$$= \frac{L.W}{2\pi} \int dk\, K\, \delta(E - \mathcal{E}(k)) \qquad (1)$$

Now $\mathcal{E}(k) = A K^n \Rightarrow d\mathcal{E} = n A K^{n-1}\, dk$

$$\Rightarrow d\mathcal{E} = n A K^{n-2}\, K\, dk$$

$$\Rightarrow K\, dK = d\mathcal{E}\, (nA)^{-1} \left(\frac{\mathcal{E}}{A}\right)^{-\frac{(n-2)}{n}} \qquad \text{since } K = \left(\frac{\mathcal{E}}{A}\right)^{1/n}$$

(1) becomes:

$$D(E) = \frac{L.W}{2\pi}\, n^{-1}.\, A^{\frac{2}{n}} \int \mathcal{E}^{-\frac{(n-2)}{n}}\, \delta(E - \mathcal{E}(k))\, d\mathcal{E}$$

$$= \frac{L.W}{2\pi} . n^{-1}\, E^{-\frac{(n-2)}{n}} \cdot A^{\frac{2}{n}}$$

* Density of Mode :

$$M(E) = \sum_{\vec{k}} \delta(E - \varepsilon(k)) \, \frac{\pi\hbar}{L} \, |v_x(k)|$$

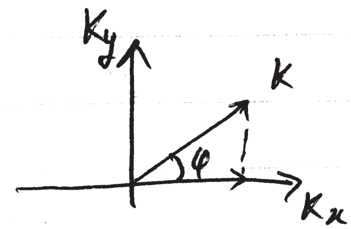
$$v_x = \frac{1}{\hbar} \frac{\partial \varepsilon(k)}{\partial k_x} \quad \text{. Then}$$

$$M(E) = \frac{L.W}{(2\pi)^2} \int_0^{2\pi} d\varphi \int_0^{\infty} k.dk \; \delta(E - \varepsilon(k)) \, \frac{\pi}{L} . \hbar \, \frac{1}{\hbar} \left| \frac{\partial \varepsilon(k)}{\partial k_x} \right|$$

$$= \frac{W}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\infty} d\varepsilon \, \delta(E - \varepsilon(k)) \, k. \left| \frac{dk}{dk_x} \right|$$

Now : $\quad k = \sqrt{k_x^2 + k_y^2} \Rightarrow \dfrac{dk}{dk_x} = \dfrac{k_x}{\sqrt{k_x^2 + k_y^2}} = \dfrac{k_x}{k}$

$$\Rightarrow k.\frac{dk}{dk_x} = k_x = k.\cos\varphi$$



and $\quad k = \left(\dfrac{\varepsilon}{A}\right)^{1/n}$

So

$$M(E) = \frac{\cancel{L}.W}{4\pi} \underbrace{\int_0^{2\pi} |\cos\varphi| d\varphi}_{= \underbrace{2 \int_{-\pi/2}^{\pi/2} \cos\varphi \, d\varphi}_{4}} \int_0^{\infty} d\varepsilon \left(\frac{\varepsilon}{A}\right)^{1/n} \delta(E - \varepsilon(k))$$

$$\boxed{M(E) = \frac{W}{\pi} \cdot \left(\frac{E}{A}\right)^{1/n}}$$

2) L is large enough:
W is small

$\Rightarrow \quad K_y = \frac{2\pi}{W} \nu \qquad \nu$ is an integer.

* Density of state:

$$D(E) = \sum_{\vec{K}} \delta(E - \mathcal{E}(k)) = \sum_{K_y} \sum_{K_x} \delta(E - \mathcal{E}(k))$$

$$= \sum_{\nu} \frac{L}{2\pi} \int_{-\infty}^{+\infty} dk_x \, \delta(E - A(K_x^2 + K_y^2)^{n/2})$$

Since $\quad \mathcal{E} = A K^n = A(K_x^2 + \underbrace{K_y^2}_{\mathcal{E}_\nu})^{\frac{n}{2}} = A(K_x^2 + \mathcal{E}_\nu)^{n/2}$

$$\Rightarrow d\mathcal{E} = \frac{n}{2} A (K_x^2 + \mathcal{E}_\nu)^{\frac{n}{2}-1} \, 2k_x \, dk_x$$

$$= \frac{n}{2} \mathcal{E} \frac{1}{K_x^2 + \mathcal{E}_\nu} 2K_x \, dk_x \qquad *$$

And using $\quad K_x = \sqrt{\left(\frac{\mathcal{E}}{A}\right)^{\frac{2}{n}} - \mathcal{E}_\nu}$

$$* \Rightarrow dk_x = d\mathcal{E} \frac{(K_x^2 + \mathcal{E}_\nu)}{n . \mathcal{E} K_x} = \frac{d\mathcal{E}}{n\mathcal{E}} \left(\frac{\mathcal{E}}{A}\right)^{\frac{2}{n}} \frac{1}{\sqrt{\left(\frac{\mathcal{E}}{A}\right)^{\frac{2}{n}} - \mathcal{E}_\nu}}$$

And change the $\int_{-\infty}^{+\infty} dk_x$ to $2\int_{\mathcal{E}_\nu}^{\infty} d\mathcal{E}$

put back to the density of states:

$$D(E) = \sum_{\nu} \frac{L \cdot 2}{2\pi} \int_{\varepsilon_\nu}^{\infty} \frac{1}{n\varepsilon} \left(\frac{\varepsilon}{A}\right)^{\frac{2}{n}} \frac{1 \cdot \delta(E-\varepsilon)}{\sqrt{\left(\frac{\varepsilon}{A}\right)^{\frac{2}{n}} - \varepsilon_\nu}} \, d\varepsilon$$

$$= \sum_{\nu} \frac{L}{\pi \cdot n} \frac{1}{A^{2/n}} \frac{E^{\frac{2}{n}-1}}{\sqrt{\left(\frac{E}{A}\right)^{\frac{2}{n}} - \varepsilon_\nu}}$$

* Density of modes:

$$M(E) = \sum_{\vec{k}} \delta(E - \varepsilon(k)) \frac{\pi\hbar}{L} |v_x(x)|$$

$$= \sum_{\nu} \frac{L}{2\pi} \int_{-\infty}^{+\infty} dk_x \, \delta(E - \varepsilon(k)) \frac{\pi\hbar}{L} \cdot \frac{1}{\hbar} \left| \frac{\partial \varepsilon(k)}{\partial k_x} \right|$$

change $\int_{-\infty}^{+\infty} dk_x = 2 \int_{0}^{+\infty} dk_x$

$$= \sum_{\nu} \frac{L}{\pi} \int_{0}^{+\infty} dk_x \, \delta(E - \varepsilon(k)) \cdot \frac{\pi}{L} \frac{d\varepsilon(k)}{dk_x}$$

$$= \sum_{\nu} \int_{\varepsilon_\nu}^{\infty} d\varepsilon \, \delta(E - \varepsilon(k)) = \sum_{\nu} \theta(E - \varepsilon_\nu)$$