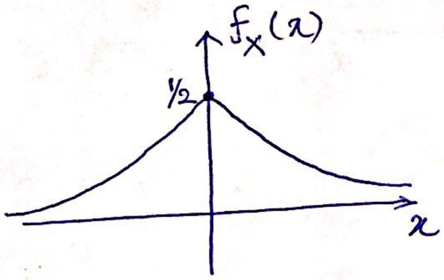
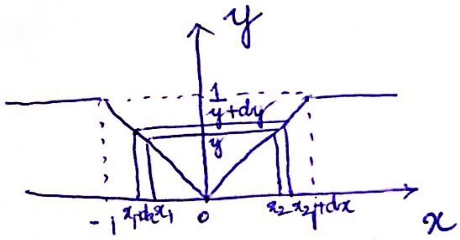
(1) $f_X(x) = \frac{1}{2} e^{-|x|}$



$$Y = g(X) \text{ where } g(x) = \begin{cases} |x| , & -1 \le x \le 1 \\ 1 , & \text{else} \end{cases}$$



$$P(y+dy \ge Y \ge y) = P(x_1+dx \le X \le x_1) + P(x_2 \le X \le x_2+dx)$$

$$f_Y(y) \cdot dy = f_X(x_1)dx + f_X(x_2)dx$$

$$f_Y(y)dy = 2f_X(x)dx \quad (\because x_1 = -x_2)$$

$$f_Y(y) = 2 f_X(x) \cdot \left|\frac{dx}{dy}\right|$$

$$f_Y(y) = e^{-|y|} \text{ for } 0 \le y \le 1$$

$$f_Y(y=1) = 1 - \int_{-1}^{1} \frac{1}{2} e^{-|x|} dx$$

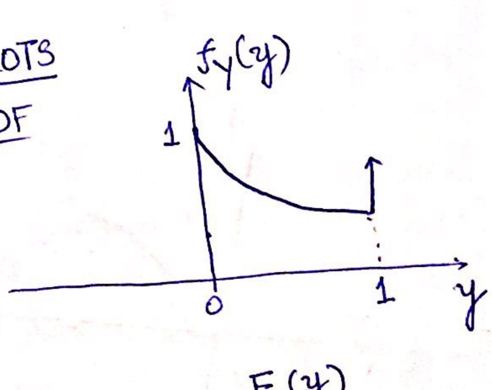
$$= 1 - \int_{0}^{1} e^{-x} dx$$

$$= 0.36788$$

$$f_Y(y) = \begin{cases} e^{-|y|}, & 0 \le y \le 1 \\ 0.36788, & y=1 \end{cases}$$

PLOTS

PDF



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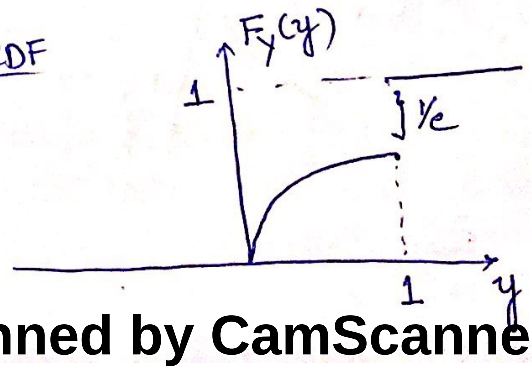
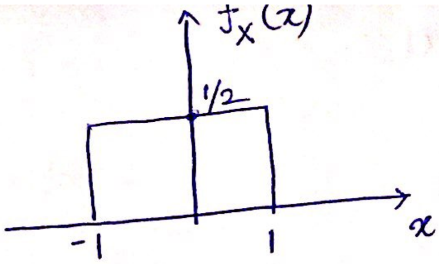


(2)

$$f_X(x) = \begin{cases} 1/2, & -1 \le x \le 1 \\ 0, & \text{else} \end{cases}$$



$Y = X^n$, n is a positive number

If n is even, range of $Y = 0 \le y \le 1$

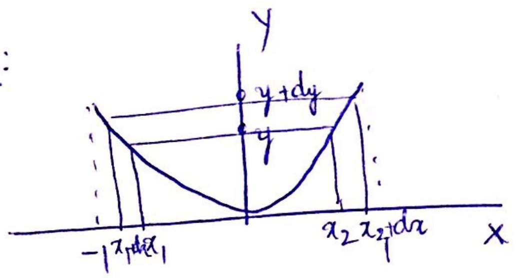
If n is odd, range of $Y = -1 \le y \le 1$

$$P(Y \le y) = P(X^n \le y) = P(X \le y^{1/n})$$

$\therefore F_Y(y) = F_X(y^{1/n})$ if n is odd

and $F_Y(y) = P(-y^{1/n} \le X \le y^{1/n})$ if n is even

If n is even:

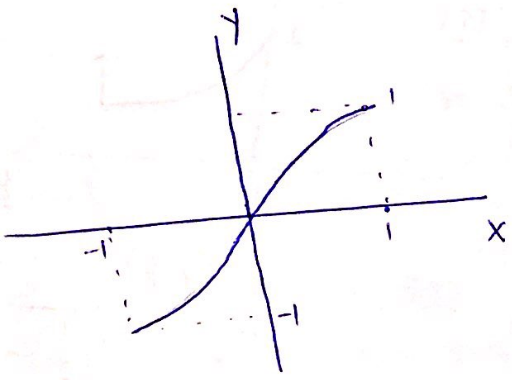


$$f_Y(y)\,dy = 2 \cdot f_X(x)\,dx$$

$$f_Y(y) = 2\, f_X(x) \left|\frac{dx}{dy}\right|$$

$$f_Y(y) = \frac{2 \times 0.5}{n\, y^{(1-1/n)}}$$

If n is odd:



$$f_Y(y)\,dy = f_X(x)\,dx$$

$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$$

$$f_Y(y) = \frac{1}{2} \cdot \frac{1}{n y^{(1-1/n)}}$$

If n is even: $F_Y(y) = \int_0^y f_Y(y)\,dy = y^{1/n}$

If n is odd: $F_Y(y) = \frac{1}{2}\{ y^{1/n} + 1 \}$

(b) $E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$

$= \int_{-\infty}^{\infty} x^n \cdot \frac{1}{2}\, 1[-1,1]\,dx$

$= \int_{-1}^{1} x^n/2\,dx = \frac{1}{2}\left\{ \frac{x^{n+1}}{n+1} \right\}_{-1}^{1} = \frac{1}{2}\left\{ \frac{1}{n+1} - \frac{(-1)^{n+1}}{n+1} \right\}$

$= \frac{1-(-1)^{n+1}}{2(n+1)}$

$= \begin{cases} 0, & n \text{ is odd} \\ \frac{1}{2(n+1)}, & n \text{ is even} \end{cases}$

$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^1 \frac{y}{n y^{(1-1/n)}}\,dy$, n even

$= \frac{1}{n}\int_0^1 y^{(1-1+1/n)}\,dy$

$= \frac{1}{n}\int_0^1 y^{1/n}\,dy$

$= \frac{1}{n}\left\{ \frac{y^{1/n+1}}{1/n+1} \right\}_0^1$

$= \frac{1}{n}\left\{ \frac{1}{1/n+1} \right\}$

$= \frac{1}{n+1}$

$$E(y) = \int_{-1}^{1} y\left(\frac{1}{2}\cdot\frac{1}{n y^{(1-1/n)}}\right) dy\,, \quad n \text{ is odd}$$

$$= \frac{1}{2n}\int_{-1}^{1} y^{1/n}\, dy$$

$$= \frac{1}{2n}\left\{\frac{1}{\frac{1}{n}+1}\left(1^{1/n+1} - (-1)^{1/n+1}\right)\right\}$$

$$= 0$$

(3) $p_M(m) = (1-p)^{m-1} p\,, \quad m = 1, 2, \dots \quad 0 < p < 1$

(a) $$E(M) = \sum_{m=1}^{\infty} m p (1-p)^{m-1}$$

$$= p \sum_{m=1}^{\infty} m (1-p)^{m-1}$$

$$= p \sum_{m=1}^{\infty}\left[\left(\sum_{k=1}^{m} 1\right)(1-p)^{m-1}\right] = p \sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}(1-p)^{m-1}\right)$$

$$= p \sum\sum_{1\le k\le y\le\infty} (1-p)^{m-1} = p\sum_{k=1}^{\infty}\left(\sum_{m=k}^{\infty} q^{m-1}\right) \quad \text{where } q = 1-p$$

$$= p\sum_{k=1}^{\infty}\left(\sum_{m=k}^{\infty} q^{k-1} q^{m-k}\right) = p\sum_{k=1}^{\infty} q^{k-1}\left(\sum_{m=k}^{\infty} q^{m-k}\right)$$

$$= p\sum_{k=1}^{\infty} q^{k-1}\left(\sum_{j=0}^{\infty} q^{j}\right) = p\sum_{k=1}^{\infty} q^{k-1}\cdot\left(\frac{1}{1-q}\right)$$

$$= \cancel{p}\,\frac{1}{\cancel{p}}\cdot\sum_{k=1}^{\infty} q^{k-1} = \sum_{j=0}^{\infty} q^{j} = \frac{1}{1-q} = \frac{1}{p}$$

Variance:

$$E[m(m-1)] = \sum_m m(m-1)p(m) = \sum_{m=1}^{\infty} m(m-1)q^{m-1}p$$

$$= p\sum_{m=2}^{\infty} m(m-1)q^{m-1}$$

$$= p\sum_{m=1}^{\infty}\left[\left(2\sum_{k=1}^{m-1} k\right)q^{m-1}\right]$$

$$= 2p\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m-1} kq^{m-1}\right)$$

$$= 2p\sum\sum_{1\le k<y<\infty} kq^{m-1} = 2p\sum_{k=1}^{\infty}\left(\sum_{m=k+1}^{\infty} kq^{m-1}\right)$$

$$= 2p\sum_{k=1}^{\infty}\left[kq^k\left(\sum_{m=k+1}^{\infty} q^{m-k-1}\right)\right]$$

$$= 2p\sum_{k=1}^{\infty}\left[kq^k\left(\sum_{j=0}^{\infty} q^j\right)\right] \; ; \; j = m-k-1$$

$$= 2p\sum_{k=1}^{\infty}\left(kq^k\frac{1}{1-q}\right) = \cancel{2p}\cdot\frac{1}{\cancel{p}}\sum_{k=1}^{\infty} kq^k$$

$$= 2\sum_{k=1}^{\infty} kq^k = \frac{2q}{p}\cdot\frac{1}{p} = \frac{2q}{p^2}$$

$$E(m^2) = E(m(m-1)) + E(m) = \frac{2q+p}{p^2}$$

$$var(m) = E(m^2) - E(m)^2 = \frac{2q+p}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

(b) $g(m) = (m-k)\,u(m-k) \Rightarrow g(m) = \begin{cases} 0, & m<k \\ m-k, & m \geq k \end{cases}$

$N = g(M)$

for $m \geq k$, $N = m-k$

$\Rightarrow m = N+k$

$\therefore p_N(n) = (1-p)^{n+k-1}\,p$ where $n+k = 1, 2, \ldots$

$\Rightarrow n = 1-k, 2k, \ldots$

(c) $E[N] = \sum_{n=1-k}^{\infty} n\,p(1-p)^{n+k-1}$

$n+k=z \quad = \sum_{z=1}^{\infty} (z-k)\,p(1-p)^{z-1}$

$= p\left\{\sum_{z=1}^{\infty}\sum_{k=1}^{z}(1-p)^{z-1} - \sum_{z=1}^{\infty}\sum_{j=1}^{k}(1-p)^{z-1}\right\}$

$= p\left\{\frac{1}{p^2} - \sum_{z=1}^{\infty}(1-p)^{z-1}k\right\}$

$= p\left\{\frac{1}{p^2} - k\frac{1}{p}\right\} = \frac{1}{p} - k$

$E[N^2] = \sum_{z=1}^{\infty}(z-k)^2\,p(1-p)^{z-1}$

$= \sum_{z=1}^{\infty}(z^2+k^2-2zk)\,p(1-p)^{z-1}$

$= p\sum_{z=1}^{\infty} z^2(1-p)^{z-1} + pk^2\sum_{z=1}^{\infty}(1-p)^{z-1} - 2k\sum_{z=1}^{\infty} pz(1-p)^{z-1}$

$$= \frac{2q}{p^2} + \frac{p}{p^2} + \cancel{p}k^2 \times \frac{1}{\cancel{p}} - \frac{2k}{p}$$

$$E[N^2] = \frac{2q+p+k^2p^2-2kp}{p^2}$$

$$\text{variance}(N) = E[N^2] - E[N]^2 = \frac{2q+p+k^2p^2-2kp}{p^2} - \left(\frac{1}{p} - k\right)^2$$