MA598A Summer 2013

Problem Set 1: Group Theory I

(1) (a) Define *normal subgroup*. Be as succinct as possible.
   (b) Let G be a finite group with $|G| = n$, and suppose H is a subgroup of G with $|G : H| = p$, with p the smallest prime divisor of n. Show that H is normal in G.

(2) Suppose that $(V, < \, , >)$ is an inner product space.
   True or false: the set of isometries of V (i.e. the set of automorphisms preserving the inner product) is a proper subgroup of $\mathrm{Aut}(V)$.

(3) (a) Define what it means for a group to be *simple*.
   (b) True or false: an abelian group is simple if and only if it is cyclic.
   (c) Let S_n denote the symmetric group on n letters.
      - A_n, the set of odd permutations in S_n, is a normal subgroup.
      - Assume that A_5 is simple. Show that A_n is simple for all $n \geq 5$.
   (d) Show that any group of order p^2q, p, q prime, is not simple.

(4) Show that any group G for which $\mathrm{Aut}(G)$ is cyclic must be abelian.

(5) Let $D_4 = \langle (1234), (12)(34) \rangle \subset S_4$ be the Dihedral group of order 8.
   (a) Show that D_4 has exactly three subgroups of order 4.
   (b) Show that exactly one of these subgroups is cyclic.
   (c) Show that the intersection of these subgroups is the commutator subgroup of D_4.

(6) Let G be a finite group and H a proper subgroup of G. Show that there exists an element of G which is not conjugate to any element of H. Does this remain true if G is allowed to be infinite?

(7) Let G be a group of odd order. Show that if $g \in G - \{e\}$, then g is not conjugate to its inverse.

(8) Determine the number of pairwise non-isomorphic groups of order 15.

(9) Suppose that G is a group and

$$\phi \in \mathrm{Hom}(G), \quad g \mapsto g^2,$$

is a homomorphism.
   - Show that G is abelian.
   - Show that there exists a finite nonabelian group where $\phi(g) = g^4$ *is* a homomorphism.

(10) Suppose

$$\varphi : G \times G \to G$$

is a homomorphism and that there exists $n \in G$ such that

$$\varphi(n, g) = \varphi(g, n) = g, \quad \text{for all } g \in G.$$

Show that G is abelian and φ is simply group multiplication.

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