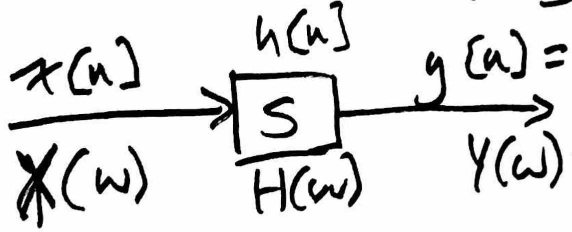
Convolution Property

Aug 1

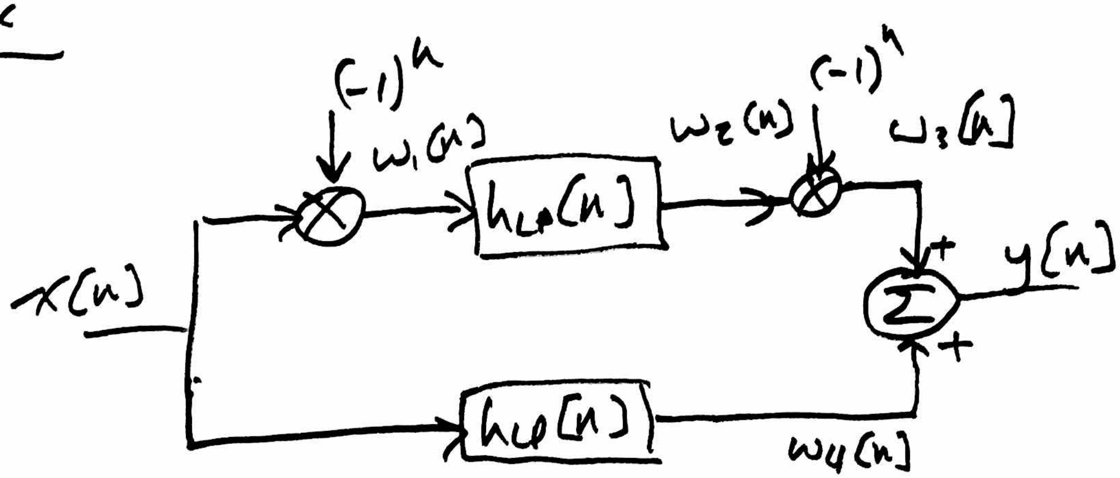


$y[n] = x[n] * h[n]$

$Y(\omega) = X(\omega) H(\omega)$

→ Just like the CTFT.

Ex



$$H_{LP}(\omega) = \begin{cases} 1 & |\omega| < \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| < \pi \end{cases}$$

periodic with period 2π

Upper branch:

$$w_1[n] = x[n] \cdot (-1)^n = x[n] e^{j\pi n} \xleftrightarrow{DTFT} X(\omega - \pi)$$

$$w_2[n] = w_1[n] * h_{LP}[n] \xleftrightarrow{DTFT} X(\omega - \pi) \cdot H_{LP}(\omega)$$

$$w_3[n] = w_2[n] e^{j\pi n} \xleftrightarrow{DTFT} X(\omega - 2\pi) H_{LP}(\omega - \pi)$$

$$= X(\omega) H_{LP}(\omega - \pi)$$

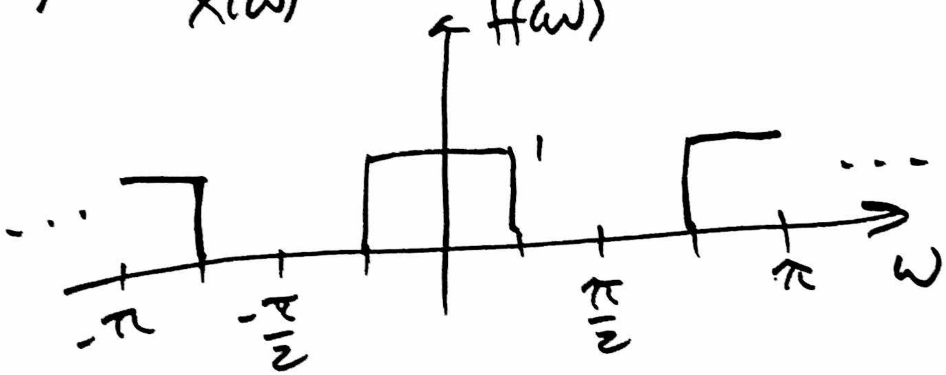
Lower branch:

$$w_4[n] = x[n] * h_{LP}[n] \xleftrightarrow{DTFT} X(\omega) H_{LP}(\omega)$$

$$y[n] = w_3[n] + w_4[n]$$

$$Y(\omega) = X(\omega) H_{LP}(\omega - \pi) + X(\omega) H_{LP}(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = H_{LP}(\omega - \pi) + H_{LP}(\omega)$$



Key points:

- Usually easier to analyze these systems in the frequency domain.
- Break a system into small steps.

Multiplication Property

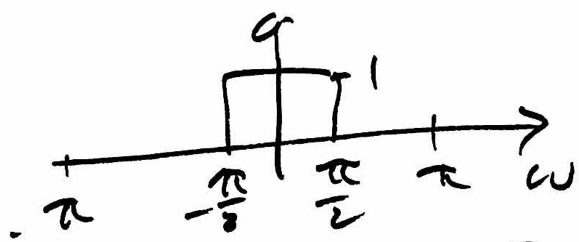
$$x_1[n] \cdot x_2[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\theta) X_2(\omega-\theta)\, d\theta \rightarrow \text{periodic convolution}$$

$$= \frac{1}{2\pi} X_1(\omega) \circledast X_2(\omega)$$

We can perform periodic by looking at a single period of $X_1(\omega)$ & $X_2(\omega)$ and use linear convolution.

Ex

$$X_1(\omega) = \begin{cases} 1, & |\omega| < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < |\omega| < \pi \end{cases} \quad \text{periodic with period } 2\pi$$

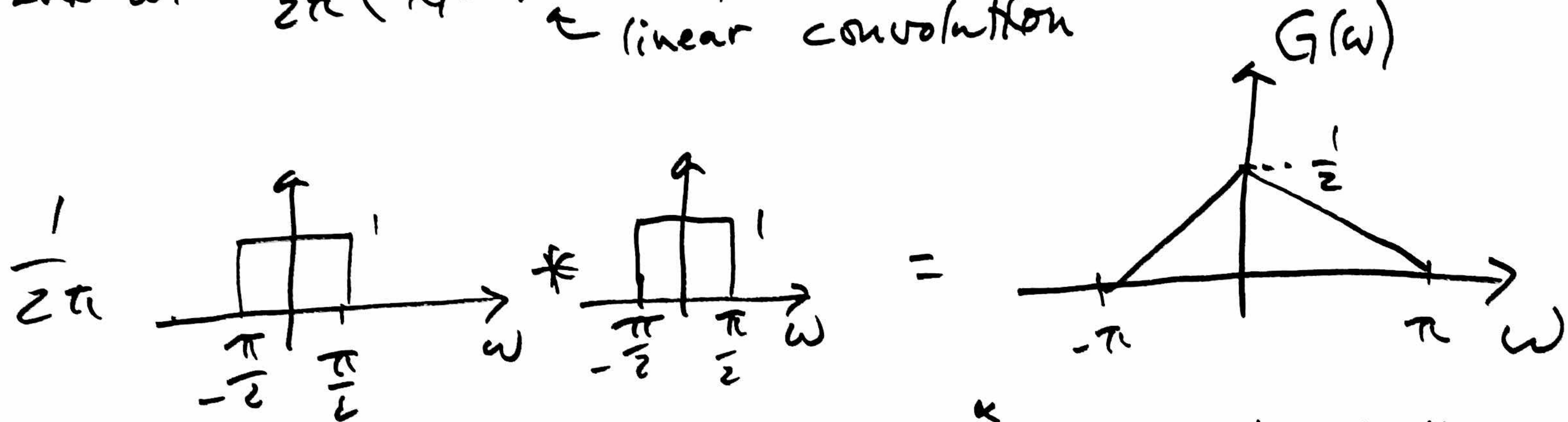


What is the DTFT of $x_1^2[n]$?

$$x_1[n] \cdot x_1[n] \xleftrightarrow{DTFT} \frac{1}{2\pi}\left(X_1(\omega) \circledast X_1(\omega)\right)$$

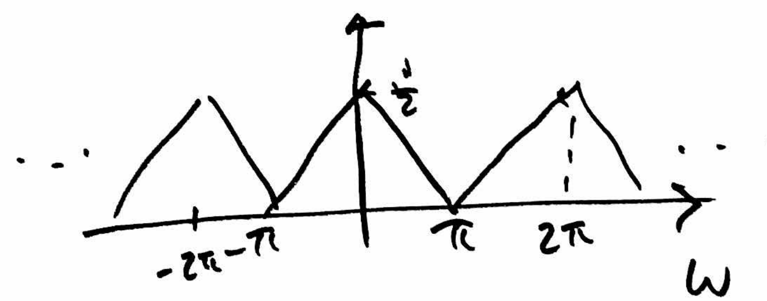
Let $\hat{X}_1(\omega) = \begin{cases} X_1(\omega), & -\pi < \omega < \pi \\ 0 & \text{else} \end{cases}$

Look at $\frac{1}{2\pi}\left(\hat{X}_1(\omega) * \hat{X}_1(\omega)\right) = G(\omega)$ ← linear convolution



Take $G(\omega)$, fold it around $\pm\pi$ ~~to~~ make it periodic.

$\frac{1}{2\pi}\left(X_1(\omega) \circledast X_1(\omega)\right)$

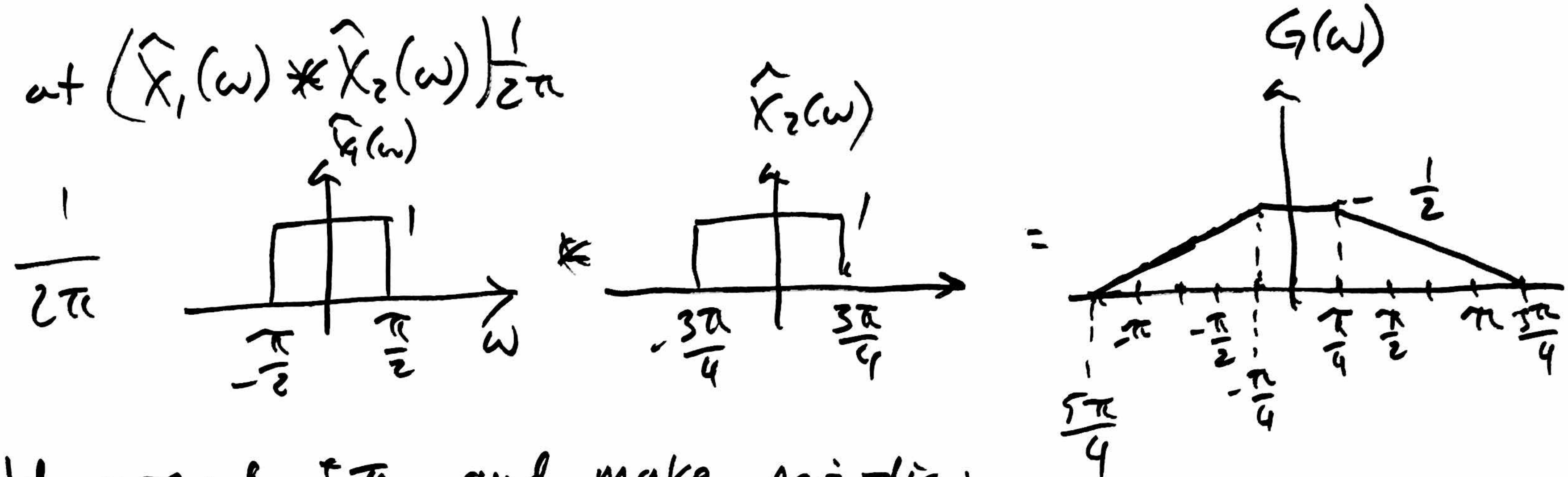


Ex

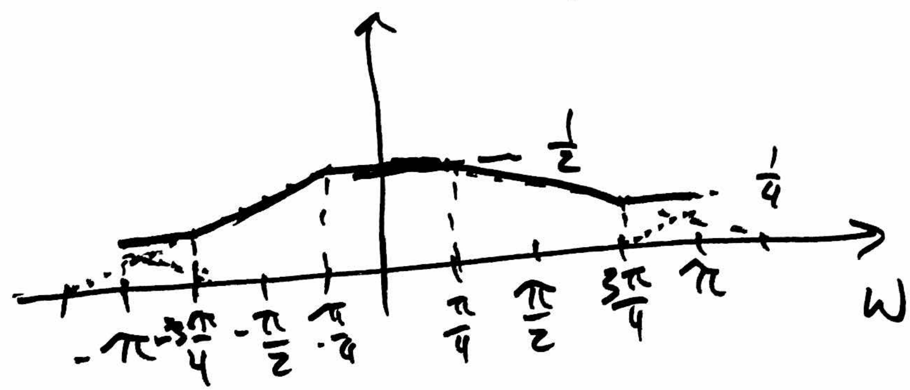
$X_1(\omega)$ as before

$$X_2(\omega) = \begin{cases} 1, & |\omega| < \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} < |\omega| < \pi \end{cases}$$ periodic with period 2π

Look at $\left(\hat{X}_1(\omega) * \hat{X}_2(\omega)\right)\frac{1}{2\pi}$



Fold around $\pm\pi$ and make periodic:



periodic with period 2π