

## Assignment 6: Absolute Continuity and Differentiation

1. Construct absolutely continuous functions  $f : [a, b] \rightarrow [c, d]$  and  $g : [c, d] \rightarrow \mathbf{R}$  such that  $g \circ f$  is not AC.
2. Suppose  $f \in L^1([a, b])$  and

$$\int_a^c f(t)dt = 0 \text{ for all } c \in [a, b].$$

Show that  $f = 0$  a.e.

3. Does there exist a strictly increasing function  $f$  defined on an interval  $I$  such that  $f' = 0$  a.e. on  $I$ ?
4. Let  $I$  and  $J$  be bounded intervals,  $f : I \rightarrow J$  AC and  $\phi : J \rightarrow \mathbf{R}$  Lipschitz. Show that  $\phi \circ f$  is AC on  $I$ .
5. Let  $f : [0, 1] \rightarrow \mathbf{R}$  be  $C^1$ , and let

$$Z = \{x \in [0, 1] : f'(x) = 0\}.$$

Show that  $\mu\{f(Z)\} = 0$ .

6. Let  $g : [0, 1] \rightarrow \mathbf{R}$  be continuous,  $g(y) \leq g(x)$  if  $y \leq x$ .
  - (a) Suppose  $g'(x) = 0$  for all but perhaps finitely many  $x$ . Show that  $g$  is constant.
  - (b) Suppose  $g'(x) = 0$  for all but perhaps countably many  $x$ . Show that  $g$  is constant.
7. True or false: There exists a measurable set  $E$  satisfying  $|E \cap I| = c|I|$ ,  $0 < c < 1$  for all bounded intervals  $I$ .
8. Let  $\{f_n\} \in AC(I)$ ,  $I = [a, b]$ . Assume  $f_n \rightarrow f$  in  $L^1$  and  $\{f'_n\}$  is Cauchy in  $L^1$ . Show there exists  $g \in AC(I)$  with  $g = f$  a.e.
9. Suppose  $f, f' \in L^1(\mathbf{R})$ , and suppose that given any bounded interval  $I$ ,  $f \in AC(I)$ . Show that
  - (a)  $\int_{\mathbf{R}} f' = 0$ .
  - (b)  $\lim_{|x| \rightarrow \infty} f(x) = 0$
  - (c) Find a counterexample if the hypothesis of absolute continuity is dropped.