MA544 Qual Practice Exam 4 Bridges Summer 08

1. Let $\pi + \mathbb{Q} = \{\pi + q : q \in \mathbb{Q}\}$. Let $\mathcal{A}$ be the σ-algebra generated by the sets $\{x\}; x \in \pi + \mathbb{Q}$. Without justification find all possible values for the Lebesgue measure, $|A|$ if $A \in \mathcal{A}$.

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, and let G be an open set in $\mathbb{R}^n$. Proof or counter-example: $f(G) = \{y | \exists x \in G \text{ with } f(x) = y\}$ is measurable.

3. Let f_n be a sequence of functions in $L^p, 1 \leq p < \infty$, which converge almost everywhere to a function f in L^p. Show that $f_n \to f$ in L^p iff $||f_n||_p \to ||f||_p$.

4. Let f be a nonnegative Lebesgue measurable function on $(0, \infty)$ such that f^2 is integrable. Let $F(x) = \int_0^x f(t)dt$ where $x > 0$. Show that
$$\lim_{x \to 0^+} \frac{F(x)}{\sqrt{x}} = 0.$$

5. Suppose that $\{f_n\}_{n \geq 1}$ is an equicontinuous family of functions on a compact set K such that the family of functions is point-wise bounded.
   (a) Show that $\{f_n\}_{n \geq 1}$ is uniformly bounded on K.
   (b) Show that $f(x) = \inf\{f_n(x) : n \geq 1\}$ is uniformly continuous on K.

6. Suppose f is an integrable function. Prove that
$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) \cos(nx) dx = 0.$$