MA598 - Complex Analysis Qual Prep - Summer 2014
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Assignment 1

6/17/2014

1. Show that an uncountable subset of the complex plane must have a limit point. Use this to prove that a non-constant holomorphic function has at most countably many zeroes.
2. Suppose that $\Omega \subset \mathbb{C}$ is a domain and that $f : \Omega \to \mathbb{C}$ is a continuous function. If f^2 and f^3 are both analytic, show that f is as well. Does this still hold if the assumption of continuity is dropped?
3. Suppose that f is analytic on the closed unit disk. Show that f is even if and only if its Taylor expansion at $z = 0$ contains only even powers of z.
4. Let f and g be two analytic functions on a domain D, and suppose that
$$f(z)\,\overline{g(z)}$$
is real-valued. Prove that $f = c\,g$, where c is a real constant, or that $g = 0$.
5. Find the radius of convergence for the power series.
   (a)
   $$\sum_{n=1}^{\infty} \frac{n^n}{n!} z^{2n}$$
   (b)
   $$\sum_{n=1}^{\infty} \frac{z^{n^2}}{n!}$$
   (c)
   $$\sum_{n=1}^{\infty} \frac{z^n}{(1+2i)^n}.$$
6. Suppose that f is analytic in a neighborhood of 0 and that
$$\sum_{n=0}^{\infty} f^{(n)}(0)$$
converges. Show that f extends to an entire function.
7. Suppose that
$$f(z) := \sum_{n=0}^{\infty} c_n z^n$$
has radius of convergence R, with $0 < R \leq \infty$. Let B_R denote the open ball of radius R, with the convention that $B_\infty = \mathbb{C}$.

(a) Show that f satisfies the Cauchy-Riemann equations on B_R. Carefully justify your steps.

(b) For each of the following, find an example or prove that none exists. Treat the cases $R < \infty$ and $R = \infty$ separately.

   i. f is bounded on B_R.

   ii. f extends to a continuous function on the closure of B_R. (if $R = \infty$, take this to mean that the limit exists as $z \to \infty$.)

   iii.
   $$\lim_{|z| \to R} |f(z)| \quad \text{exists.}$$

   iv. f is surjective.

   v. (here only treat $R < \infty$) An f which extends to a bounded, continuous function on $\mathbb{C}$, which is analytic for $|z| \neq R$.

(c) If $R = \infty$, characterize the topological closure of the image of f.

8. Let f be analytic near 0 and satisfy

$$f'(z) = qf(q^2 z), \quad f(0) = 1,$$

where $q \in \mathbb{R}$ is a constant with $0 < q < 1$. Determine the radius of convergence of the Taylor series for f centered at 0.

9. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is continuous (not necessarily holomorphic) and let $z_0 \in \mathbb{C}$. Show that

$$\lim_{r \to 0^+} \int_{C_r(z_0)} \frac{f(z)}{z - z_0} \, dz = 2\pi i \, f(z_0).$$