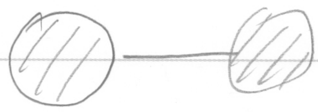
4(a) Interiors of connected sets are not always connected. Counter eg:

$$A = B_1(-2) \cup B_1(2) \cup \{(x,0) : -1 \le x \le 1\} \subset \mathbb{R}^2$$



A

So $A^\circ = B_1(-2) \cup B_1(2)$ not connected, clearly.

(b) We prove closures of connected sets are still connected.

Let A be any connected set in a metric space (X,d).

Suppose (by contradiction) $\bar{A}$ is not connected.

$\Rightarrow \bar{A} = B \cup C$ w/ $\bar{B} \cap C = \bar{C} \cap B = \emptyset$, $B \neq \emptyset$, $C \neq \emptyset$.

Now $\Rightarrow A \subset (B \cup C)$. If $A \cap B$ & $A \cap C$ are both non-empty we see $A = (A \cap B) \cup (A \cap C)$ &

$$\left.\begin{aligned} (\overline{A\cap B} \cap A \cap C) &\subset \bar{B} \cap C = \emptyset \\ (A \cap B \cap \overline{A \cap C}) &\subset B \cap \bar{C} = \emptyset \end{aligned}\right\} \Rightarrow A \text{ is separated } ※$$

Contradiction, so indeed $A \subset B$ or $A \subset C$.

WLOG $A \subset B$.

$\Rightarrow \exists \; x \in A' \cap C$ (Since $C \neq \emptyset$ & $\bar{A} = B \cup C$.)

But $\Rightarrow \forall \; \varepsilon > 0 \;\; N_\varepsilon(x) \cap B \supseteq N_\varepsilon(x) \cap A \neq \emptyset$

$\Rightarrow x \in \bar{B} \Rightarrow C \cap \bar{B} \neq \emptyset$ ※ contradiction.

So $\bar{A}$ still connected $\blacksquare$