Jeremy Garcia **Math 598 Week 3** **June 25, 2013**

Problem 101.
(a) Show that $x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Z}_3[x]$.
(b) Show that $x^4 + 1$ is not irreducible in $\mathbb{Z}_3[x]$.

Proof.
(a) First note that $f(x) = x^4 + x^3 + x^2 + x + 1$ cannot be reduced into the product of a linear and cubic polynomial since then $f(x)$ would have a root in $\mathbb{Z}_3$. This can be seen by $f(0) = 1$, $f(1) = 2$, and $f(2) = 1$. Now, if $f(x)$ is to be reduced, then it must factor into the product of two irreducible quadratic polynomials over $\mathbb{Z}_3$. The only quadratic irreducible polynomials over $\mathbb{Z}_3$ are:

$$\begin{array}{ll} x^2 + 1 & 2x^2 + 2 \\ x^2 + x + 2 & 2x^2 + x + 1 \\ x^2 + 2x + 2 & 2x^2 + 2x + 1 \end{array}$$

Now since the polynomial $f(x)$ is monic and has constant term 1, if $f(x)$ were to factor into the product of two quadratic irreducibles, then these two irreducibles would both need to have the same leading coefficient and same constant term since every nonzero square in $\mathbb{Z}_3$ is 1. Thus, we are left with eight computations to make by brute force, in which we will get part (b) for free.

$$\begin{aligned}
&(x^2+1)(x^2+1) = x^4 + 2x^2 + 1 \\
&(x^2+x+2)(x^2+x+2) = x^4 + 2x^3 + 2x^2 + x + 1 \\
&(x^2+2x+2)(x^2+2x+2) = x^4 + x^3 + 2x^2 + 2x + 1 \\
&(x^2+x+2)(x^2+2x+2) = x^4 + 1
\end{aligned}$$

$$\begin{aligned}
&(2x^2+2)(2x^2+2) = x^4 + 2x^2 + 1 \\
&(2x^2+x+1)(2x^2+x+1) = x^4 + x^3 + 2x^2 + 2x + 1 \\
&(2x^2+2x+1)(2x^2+2x+1) = x^4 + 2x^3 + 2x^2 + x + 1 \\
&(2x^2+x+1)(2x^2+2x+1) = x^4 + 1
\end{aligned}$$

Hence, $f(x) = x^4 + x^3 + x^2 + x + 1$ cannot be factored into the product of quadratics over $\mathbb{Z}_3$ and is therefore irreducible in $\mathbb{Z}_3[x]$.

(b) From part (a), we have $x^4 + 1 = (x^2 + x + 2)(x^2 + 2x + 2)$. Therefore, $x^4 + 1$ is not irreducible in $\mathbb{Z}_3[x]$.

□