LAB #9

Predator-Prey Problems

Goal: Investigate the interaction of species via a particular predator-prey problem.

Required tools: MATLAB routines ***pplane*** , ***dfield*** and `fplot`.

Discussion

You will examine a predator-prey problem that has historical roots as noted in the following excerpt of an article.

Predator-Prey Problem:
Why the Percentage of Sharks Caught in the Mediterranean Sea
Rose Dramatically During World War I

In the mid 1920's the Italian biologist Umberto D'Ancona was studying the population variations of various species of fish that interact with each other. During his research he came across some data on percentages-of-total-catch of several species of fish that were brought into different Mediterranean ports in the years that spanned WWI. The data gave the percentage-of-total-catch of selachians (sharks, skates, rays, etc) which are not very desirable as food fish. The data for the port of Fiume, Italy, during 1914-1923 was as follows:

1914	**1915**	**1916**	**1917**	**1918**	**1919**	**1920**	**1921**	**1922**	**1923**
11.9%	21.4%	22.1%	21.2%	36.4%	27.3%	16.0%	15.9%	14.8%	10.7%

D'Ancona was puzzled by the very large increase in the percentage of selachians during the period of the war. Obviously, he reasoned, the increase in the percentage of selachians was due to the greatly reduced level of fishing during this period. But how does the intensity of fishing affect the fish population ? The answer to this question was of great concern to D'Ancona in his research on the struggle for existence between competing species. It was also of concern to the fishing industry, since it would have obvious implications for the way fishing should be done.

What distinguishes the selachians from the food fish is that the selachians are **predators**, while the food fish are their **prey**; the selachians depend on the food fish for their survival. At first, D'Ancona thought that this accounted for the large increase of selachians during the war. Since the level of fishing was greatly reduced during this period, there were more prey available to the selachians, who therefore thrived and multiplied rapidly. However, this explanation does not hold any water since there were also more food fish during this period. D'Ancona's theory only shows that there are more selachians when the level of fishing is reduced; it does not explain why a reduced level of fishing is *more* beneficial to the predators than their prey.

After exhausting all possible biological explanations of this phenomenon, D'Ancona turned to his colleage, the famous Italian mathematician Vito Volterra. Hopefully, Volterra would formulate a mathematical model of the growth of the predator (selachians) and their prey (food fish), and this model would provide the answer to D'Ancona's question. Volterra began his analysis of this problem by separating all the fish into the prey population $x(t)$ and the predator population $y(t)$. He then reasoned that the food fish do not compete very intensively among themselves since their food is generally plentiful. Hence, in the absence of the selachians, the food fish would grow according to the Malthusian growth model $x' = ax$, for some positive constant a. Next, he reasoned that the number of contacts per unit time between predators and prey is bxy, for some positive constant b. Hence $x' = ax - bxy$. Similarly, Volterra concluded that the predators have a natural rate of decrease $-cy$, proportional to their present number, and that they also increase at a rate dxy, proportional to their present number y and their food supply x. Thus,

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy = x(a - by) \\ \\ \dfrac{dy}{dt} = -cy + dxy = y(-c + dx) \end{cases} \qquad (*)$$

This system of equations governs the interaction of the selachians and food fish in the *absence* of fishing.

ASSIGNMENT

For the system $(*)$, we shall take $a = 1$, $b = 0.5$, $c = 0.75$, $d = 0.25$. Hence the system becomes

$$\begin{aligned} x' &= x(1 - 0.5y) \\ y' &= y(-0.75 + 0.25x) \end{aligned} \qquad (**)$$

You will investigate the behavior of solutions to this system and, at the end, examine what happens when fishing is permitted.

(1) Find the equilibrium points for this system. These are the points (x, y) obtained by setting $x' = y' = 0$ (see page 350).

(2) Enter the above system into ***pplane*** . Choose your scale so that the origin is at the center of the picture and the other equilibrium point is centered in the 1^{st} quadrant. Plot several orbits in each quadrant. It appears that all of the trajectories in the 1^{st} quadrant are closed curves. (You will see why shortly.) Notice that they all contain the equilibrium point. This is due to a general theorem which says that for a "nice" autonomous system, any closed orbit must contain an equilibrium point. What is the significance for the populations of food fish and selachians of the fact that the orbits are closed loops ?

(3) The ***pplane*** plot gives a plot of y against x. Explain why, as a function of x, the system $(**)$ yields

$$\frac{dy}{dx} = \frac{y(-0.75 + 0.25x)}{x(1 - 0.5y)}.$$

Use ***dfield*** (not ***pplane***) to plot some solutions to this equation for various initial data. Use the same ranges on x and y as part (2). You should find that the solution curves here are the same as the orbits in part (2) except that the top half and the bottom half of each closed orbit must be plotted separately. Can you think of a mathematical reason for this ? (Think about the fact that we are representing y as a function of x.)

(4) The equation in part (3) is a separable equation. Solve it to show that the general solution is given implicitly by

$$C = \frac{yx^{0.75}}{e^{0.5y}\,e^{0.25x}}\,.$$

(5) Let $C = \frac{1}{2}$ in the above equation. This should describe one orbit.

(a) Approximate the value of y which corresponds to $x = 1$. To do this, note that when $C = \frac{1}{2}$ the equation from part (4) is equivalent to

$$\frac{e^{0.25x}}{2x^{0.75}} = \frac{y}{e^{0.5y}}\,.$$

If $x = 1$, the value of the left side is approximately 0.6420. You can find the corresponding values of y by plotting (using `fplot`) the function on the right side of this equality and reading off the appropriate values.

(b) Find the values of y corresponding to $x = 2, 3, 4, \cdots, 10$. Explain why you typically get 2 such values (if you get any). You will eventually find that there is no y value for the given x. Approximate the largest value x_0 of x for which there is a corresponding y. (For this, a graph of the left side of the equation in (a) might be useful.) Explain why there is only one value of y corresponding to x_0.

(c) Approximate the smallest value x_1 of x for which there is a corresponding y. Explain why there is only one value of y corresponding to x_1.

The fact that y exists only for $x_1 \le x \le x_0$ says that the orbit in question lies over the interval $[x_1, x_0]$ on the x-axis. The fact that we get two y values for each x strictly between x_1 and x_0 says that the orbit has a top and a bottom. The fact that we get only one y value corresponding to x_1 and x_0 says that the top and bottom curves meet at the end. This is almost a proof that the orbit in question is a closed curve. In fact, it is not hard to complete this analysis to show that the orbit is indeed a closed curve.

(6) In this part, we continue part (5) to show why all of the orbits are closed. Define the functions $F(x)$ and $G(y)$ as follows

$$F(x) = \frac{x^{0.75}}{e^{0.25x}} \quad \text{and} \quad G(y) = \frac{y}{e^{0.5y}}\,.$$

Notice that the orbits all are described by an equation of the form

$$G(y) = \frac{C}{F(x)}\,.$$

Furthermore, if the orbit begins at a point in the 1^{st} quadrant, then $C > 0$. Why? Using `fplot`, plot the graph of $G(y)$ over the range $0 \le y \le 6$. Using your graph, explain why (typically) if there is a positive value of y for which the above equation is valid, then there will be two. These values correspond to the top and bottom of the closed orbit. Explain why $\lim_{x\to\infty} F(x) = 0$. Use this together with the graph of $G(y)$ to explain why if x is sufficiently large, then there are no values of y for which the above equation is true. Similarly, show that for x sufficiently near 0, there are no such y. This shows that the orbit lies over a closed interval on the x-axis. Thus, each orbit has a top and a bottom and lies over a closed interval in the x-axis.

(7) Let us now return to our system (∗∗). Using ***pplane*** plot x and y as functions of t for several different initial conditions from the 1^{st} quadrant. Do they appear to be periodic ? If so, does the period appear to depend upon the initial condition?

(8) It can be proved that $x(t)$ and $y(t)$ are indeed periodic. (This follows from the existence and uniqueness theorem together with the fact that the orbits are closed. Ask your classroom instructor for details.) Let T be the period. We can compute the average values of $x(t)$ and $y(t)$ by the formulas:

$$\overline{x} = \frac{1}{T}\int_0^T x(t)\,dt \quad \text{and} \quad \overline{y} = \frac{1}{T}\int_0^T y(t)\,dt\,.$$

Remarkably, we can evaluate these integrals exactly *without* knowing either $x(t)$ or $y(t)$. Give reasons for each of the steps below for evaluating $\overline{x}$ and use a similar argument to compute $\overline{y}$ for yourself:

$$\begin{aligned}
\frac{y'}{y} &= -0.75 + 0.25x \\
\frac{1}{T}\int_0^T \frac{y'(t)}{y(t)}\,dt &= \frac{1}{T}\int_0^T (-0.75 + 0.25x(t))\,dt \\
\frac{1}{T}(\ln|y(T)| - \ln|y(0)|) &= -0.75 + \frac{1}{T}\int_0^T 0.25x(t)\,dt \\
0 &= -0.75 + 0.25\overline{x} \\
3 &= \overline{x}
\end{aligned}$$

Notice that the average value of $x(t)$ does not depend upon how big the orbit is. Where, in your phase plane picture, does the point $(\overline{x}, \overline{y})$ appear ?

(9) Explain how the system

$$\begin{aligned}
x' &= x(1 - 0.5y) - \alpha x \\
y' &= y(-0.75 + 0.25x) - \alpha y
\end{aligned},$$

where α is a small constant, describes the effect of fishing on the population. Plot some orbits if $\alpha = 0.05$. Are there still closed orbits ? Compute the average selachian and food fish populations if $\alpha = 0.05$ and compare with the average computed in part (8). How does this result relate to the original question asked by D' Ancona to Volterra ?