ECE 302 Homework 7
Due August 2, 2016

Reading assignment: chapter 9, sections 9.1 - 9.4, 9.6; chapter 10, sections 10.1 - 10.3.

1. A coin is flipped n times. Let the random variable $X_i = 1$ if the ith flip is heads and $X_i = 0$ if the ith flip is tails, for $i = 1, 2, ..., n$. Let X be the number of heads flipped in n flips. Assume all flips are fair and independent.

   (a) What kind of random variable is X? Express X as a function of $X_1, X_2, ..., X_n$.

   (b) Find the mean and variance of X.

   Solution:

   (a) X is a binomial random variable. Since $X_i = 1$ when the ith flip is heads, we have that

   $$X = \sum_{i=1}^{n} X_i.$$

   (b) Since X can be expressed as a sum of the independent random variables X_i, we have that

   $$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i],$$
   $$\text{Var}[X] = \sum_{i=1}^{n} \text{Var}[X_i].$$

   Each X_i is a Bernoulli random variable with $p_{X_i}(1) = p_{X_i}(0) = 1/2$, so its mean and variance can be found as follows:

$$\begin{aligned}
\mathbb{E}[X_i] &= \sum_{x=0}^{1} x p_{X_i}(x) \\
&= \frac{1}{2} \\
\mathbb{E}[X_i^2] &= \sum_{x=0}^{1} x^2 p_{X_i}(x) \\
&= \frac{1}{2} \\
\implies \mathrm{Var}[X_i] &= \mathbb{E}[X_i^2] - (\mathbb{E}[X])^2 = \frac{1}{4}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}[X] &= \frac{n}{2} \\
\mathrm{Var}[X] &= \frac{n}{4}
\end{aligned}$$

2. A random process $X(t)$ is defined by

$$X(t) = \begin{cases} 1 & , T \leq t \leq T+1, \\ 0 & , \text{else}, \end{cases}$$

where T is a uniformly distributed random variable in the interval (0,1).

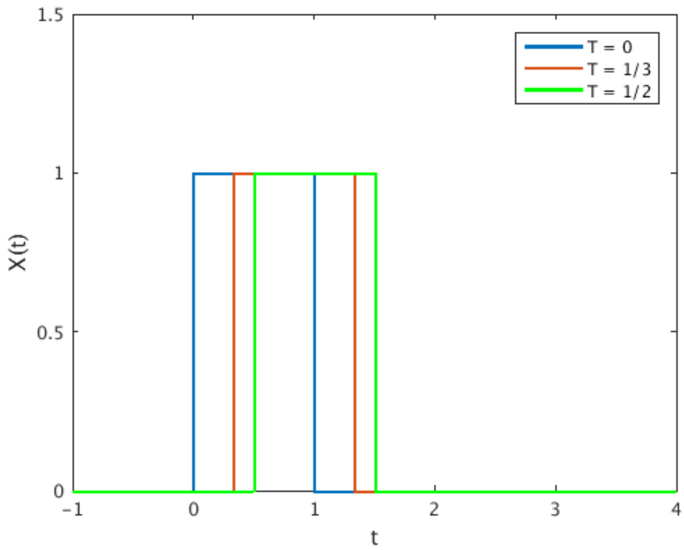
(a) Plot a few sample functions of $X(t)$.

(b) Find the pmf of $X(t)$ for a fixed value of t.

(c) Find $\mu_X(t)$ and $R_X(t_1, t_2)$. Is $X(t)$ a wide-sense stationary random process?

Solution:

(a)



(b) Note that $X(t)$ is a discrete random variable that takes on values 0 or 1 for any fixed t. Want to find $p_{X(t)}(0)$ and $p_{X(t)}(1)$ for all t.
If $t \in [0, 2]$, then

$$\begin{aligned} p_{X(t)}(1) &= \Pr(X(t) = 1) \\ &= \Pr(T \leq t \text{ and } T + 1 \geq t) \\ &= \begin{cases} \Pr(T \leq t) & , 0 \leq t < 1 \\ \Pr(T + 1 \geq t) & , 1 \leq t \leq 2 \end{cases} \\ &= \begin{cases} t & , 0 \leq t < 1 \\ 2 - t & , 1 \leq t \leq 2 \end{cases} \end{aligned}$$

If $t \notin [0,2]$, then $p_{X(t)}(0) = \Pr(X(t) = 0) = 1$.

Therefore,

$$p_{X(t)}(0) = \begin{cases} 1-t & , 0 \leq t < 1 \\ t-1 & , 1 \leq t \leq 2 \\ 1 & , \text{else} \end{cases}$$

$$p_{X(t)}(1) = \begin{cases} t & , 0 \leq t < 1 \\ 2-t & , 1 \leq t \leq 2 \\ 0 & , \text{else} \end{cases}$$

$$\begin{aligned} \mu_X(t) &= \mathbb{E}[X(t)] \\ &= \sum_{x=0}^{1} x p_{X(t)}(x) \\ &= p_{X(t)}(1) \\ &= \begin{cases} t & , 0 \leq t < 1 \\ 2-t & , 1 \leq t \leq 2 \\ 0 & , \text{else} \end{cases} \end{aligned}$$

$$\begin{aligned} R_X(t_1, t_2) &= \mathbb{E}[X(t_1)X(t_2)] \\ &= \sum_{x_2=0}^{1} \sum_{x_1=0}^{1} x_1 x_2 p_{X(t_1),X(t_2)}(x_1, x_2) \\ &= p_{X(t_1),X(t_2)}(1,1) \\ &= \Pr(X(t_1) = 1, X(t_2) = 1) \end{aligned}$$

In order to find $\Pr(X(t_1) = 1, X(t_2) = 1)$ need to consider several cases for t_1 and t_2. Will find $\Pr(X(t_1) = 1, X(t_2) = 1)$ for $t_1 \leq t_2$. The case $t_1 > t_2$ is given by symmetry.

If $0 \leq t_1, t_2 \leq 1$,

$$\begin{aligned}\Pr(X(t_1) = 1, X(t_2) = 1) &= \Pr(T \leq t_1) \\ &= t_1\end{aligned}$$

If $1 \leq t_1, t_2 \leq 2$,

$$\begin{aligned}\Pr(X(t_1) = 1, X(t_2) = 1) &= \Pr(T + 1 \geq t_2) \\ &= 2 - t_2\end{aligned}$$

If $0 \leq t_1 \leq 1 \leq t_2 \leq 2$ and $|t_1 - t_2| \leq 1$,

$$\begin{aligned}\Pr(X(t_1) = 1, X(t_2) = 1) &= \Pr(T \leq t_1 \text{ and } T + 1 \geq t_2) \\ &= t_1 - t_2 + 1\end{aligned}$$

In any other case, $\Pr(X(t_1) = 1, X(t_2) = 1) = 0$.

By symmetry, we have that

$$R_X(t_1, t_2) = \begin{cases} \min(t_1, t_2) & , 0 \leq t_1, t_2 < 1 \\ 2 - \max(t_1, t_2) & , 1 < t_1, t_2 \leq 2 \\ \min(t_1, t_2) - \max(t_1, t_2) + 1 & , 0 \leq \min(t_1, t_2) \leq 1 \leq \max(t_1, t_2) \leq 2, \\ & \quad |t_1 - t_2| \leq 1 \\ 0 & , \text{else} \end{cases}$$

Clearly, $X(t)$ is not wide-sense stationary.

3. A discrete-time random process is defined by $X(n) = A^n$, for $n \geq 0$. Assume A is a uniform random variable on the interval $(0, 1)$.

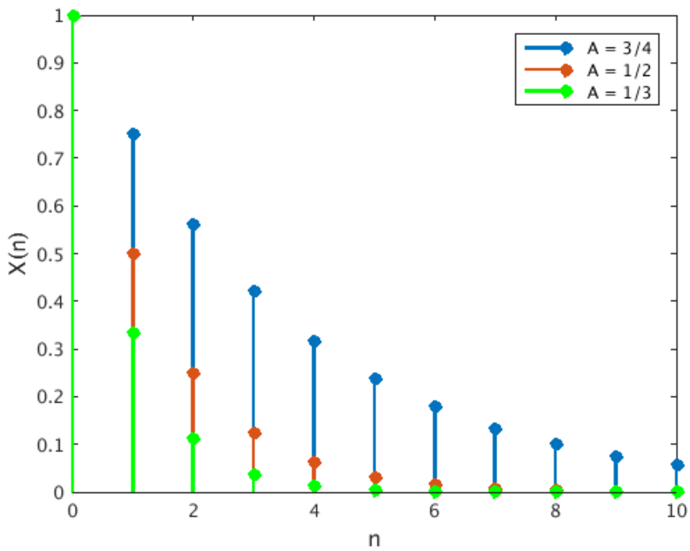
   (a) Plot a few sample functions of $X(n)$.

   (b) Find the pdf of $X(n)$ for a fixed value of n.

   (c) Find $\mu_X(n)$ and $R_X(n_1, n_2)$. Is $X(n)$ a wide-sense stationary random process?

   Solution:

   (a)



   (b) For any fixed value of $n \geq 0$, $X(n) = A^n$. The pdf of $X(n)$ can be found using the density method.

$$f_{X(n)}(x) = f_A(a)\left|\frac{dx(n)}{da}\right|^{-1}$$

$$x(n) = a^n \implies a = (x(n))^{1/n}$$

$$\frac{dx(n)}{da} = \frac{d}{da}a^n = na^{n-1} = nx^{(n-1)/n}$$

$$\implies f_{X(n)}(x) = \frac{1}{n}x^{(1-n)/n},\ x \in (0, 1)$$

(c)

$$\begin{aligned}
\mu_X(n) &= \mathbb{E}[X(n)] \\
&= \mathbb{E}[A^n] \\
&= \int_0^1 a^n da \\
&= \frac{1}{n+1}
\end{aligned}$$

$$\begin{aligned}
R_X(n_1, n_2) &= \mathbb{E}[X(n_1)X(n_2)] \\
&= \mathbb{E}[A^{n_1+n_2}] \\
&= \int_0^1 a^{n_1+n_2} da \\
&= \frac{1}{n_1+n_2+1}
\end{aligned}$$

Clearly, $X(n)$ is not wide-sense stationary.

4. Students arrive at a train station according to a Poisson process with an arrival rate of 1 student per 5 minutes.

   (a) Find the probability that the first student will arrive in the first 10 minutes.

   (b) Find the probability that the first two students will arrive in the first 10 minutes.

   (c) Find the probability that no more than two students will arrive in the first 10 minutes.

   Solution:

   (a) Let the random variable T denote the time elapsed in minutes until the arrival of the first student. Then T is an exponential random variable with $\lambda = 1/5$. Want to find $\Pr(T \leq 10)$.

   $$\begin{aligned}\Pr(T \leq 10) &= F_T(10) \\ &= 1 - e^{-10/5} \\ &= 1 - e^{-2}\end{aligned}$$

   (b) Let the random variable T_2 denote the time elapsed in minutes until the arrival of the first two students. Then T_2 is an Erlang random variable of order 2 with $\lambda = 1/5$. Want to find $\Pr(T_2 \leq 10)$.

   $$\begin{aligned}\Pr(T_2 \leq 10) &= F_{T_2}(10) \\ &= 1 - \sum_{n=0}^{1} \frac{(\frac{10}{5})^n e^{-10/5}}{n!} \\ &= 1 - 3e^{-2}\end{aligned}$$

   (c) Let the random variable K_{10} denote the number of students that arrive in the first 10 minutes. Then K_{10} is a Poisson random variable wth $\lambda = 1/5$. Want to find $\Pr(K \leq 2)$.

   $$\begin{aligned}\Pr(K_{10} \leq 2) &= \sum_{k=0}^{2} p_{K_10}(k) \\ &= \sum_{k=0}^{2} \frac{(\frac{10}{5})^k e^{-10/5}}{k!} \\ &= 5e^{-2}\end{aligned}$$

5. The input into a filter is zero-mean white Gaussian noise X(t) with noise power density $N_0/2$ W/Hz. The filter has transfer function

$$H(f) = \frac{1}{1 + j2\pi f}.$$

   (a) Find $R_X(\tau)$.
   (b) The process $X(t)$ is sampled at two time points $t_1 \neq t_2$, yielding $X(t_1)$ and $X(t_2)$. Are $X(t_1)$ and $X(t_2)$ uncorrelated?
   (c) Let $Y(t)$ be the output of the filter. Find $S_Y(f)$ and $R_Y(\tau)$. What is the average power of $Y(t)$?
   (d) Find the average power of $Y(t)$ in the frequency range $[-10, 10]$ Hz.

Solution:

(a) Want to find $R_X(\tau)$ given $S_X(f) = N_0/2$. We have that

$$S_X(f) = N_0/2 = \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}\, d\tau$$

We proceed by assuming $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$. Can show that $S_X(f) = N_0/2$.

$$\begin{aligned}
S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}\, d\tau \\
&= \int_{-\infty}^{\infty} \frac{N_0}{2}\delta(\tau)e^{-j2\pi f\tau}\, d\tau \\
&= \int_{-\infty}^{\infty} \frac{N_0}{2}\delta(\tau)e^{0}\, d\tau \\
&= \frac{N_0}{2}
\end{aligned}$$

Therefore, $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$.

(b)

$$\begin{aligned}
\text{Cov}[X(t_1)X(t_2)] &= \mathbb{E}[X(t_1)X(t_2)] - \mathbb{E}[X(t_1)]\mathbb{E}[X(t_2)] \\
&= R_X(t_1, t_2) - 0, \text{ since X(t) is zero-mean} \\
&= \frac{N_0}{2}\delta(t_2 - t_1), \text{ since X(t) is WSS} \\
&= 0, \text{ since } t_1 \neq t_2
\end{aligned}$$

Therefore $X(t_1)$ and $X(t_2)$ are uncorrelated when $t_1 \neq t_2$.

(c) From the notes, we have that

$$\begin{aligned} S_Y(f) &= \left|H(f)\right|^2 S_X(f) \\ &= \frac{N_0/2}{|1+j2\pi f|^2} \\ &= \frac{N_0/2}{1+4\pi^2 f^2} \end{aligned}$$

In order to find $R_Y(\tau)$, can use partial fractions to express $S_Y(f)$ as

$$\begin{aligned} S_Y(f) &= \frac{N_0/4}{1+j2\pi f} + \frac{N_0/4}{1-j2\pi f} \\ \implies R_Y(\tau) &= \mathcal{F}^{-1}\{S_Y(f)\} = \frac{N_0}{4}e^{-\tau}u(\tau) + \frac{N_0}{4}e^{\tau}u(-\tau), \text{ from transform tables} \\ &= \frac{N_0}{4}e^{-|\tau|}. \end{aligned}$$

The average power of the output is $\mathbb{E}[Y^2(t)] = R_Y(0) = \frac{N_0}{4}$ W.

(d) The average power of the output in the frequency range $B = [-10, 10]$ Hz can be found as

$$\begin{aligned} P_B &= \int_{-10}^{10} S_Y(f)\, df \\ &= \int_{-10}^{10} \frac{N_0/2}{1+4\pi^2 f^2}\, df \\ &= \int_{-20\pi}^{20\pi} \frac{N_0/2}{2\pi(1+u^2)}\, du \\ &= \frac{N_0 \tan^{-1}(20\pi)}{2\pi} \approx 0.2478 N_0 \end{aligned}$$