Mock Qual #2

<u>Instructions</u>: *Exam to be taken under qual settings. Upon completion, please hand in with instructions for how the graded exam should be returned.*

Complete five of the following.

1. (a) Give full statements of the following results. In case of ambiguity, provide the most general statement used in class.
   i. Cauchy's theorem
   ii. Residue theorem
   iii. Schwarz lemma
   iv. Rouché's theorem

   (b) Sketch the proof of one of the results in part (a).

2. Suppose $u : \mathbb{C} \to \mathbb{R}$ is a non-constant harmonic function. Show that the zero-set
$$\{z \in \mathbb{C} \mid u(z) = 0\}$$
is unbounded as a subset of $\mathbb{C}$.

3. Let H denote the half-plane
$$H = \{x + iy \in \mathbb{C} : y > \sqrt{2}/2\}.$$
Explain how to construct a one-to-one conformal mapping from $D_1(0) \cap H$ onto $D_1(0)$.

4. Let
$$f(z) = \sum_{n=0}^{\infty} z^{n!}\,.$$
Show that the radius of convergence of this power series is 1. Let u be a root of unity. Show that
$$\lim_{r \to 1^-} f(ru) = \infty.$$
Let $\Omega_\epsilon = D_1(0) \cup D_\epsilon(1)$. Is there $\epsilon > 0$ and a meromorphic function F such that $F = f$ on the unit disk? Explain.

5. Evaluate the integral
$$\int_0^\infty \frac{\sqrt{x}}{x^2 + 1}\, dx$$
using techniques from complex variables.

6. Suppose f is analytic on $D_1(0)$ and $|f(z)| < 1$ for all z in the unit disk. Prove that if $f(0) = a \neq 0$, then f has no zeros on $D_{|a|}(0)$.

7. Suppose f is a *one-to-one* entire function. Show that there exist $a \in \mathbb{C} - \{0\}$, $b \in \mathbb{C}$ such that $f(z) = az + b$.