2^{nd} February 2012

Q: what is $E(X)$ if

S = permutation of n elements

$X(\sigma)$ = # of inversions in σ

$\rightarrow$ (i,j) is an inversion if $i<j$ but $\sigma(i) > \sigma(j)$

Let $X_{ij}(\sigma) = \begin{cases} 1 & \text{if } (i,j) \text{ is inversion} \\ 0 & \text{else} \end{cases}$

Then, $X(\sigma) = \sum_{i<j} X_{ij}(\sigma)$

We want to know $E(X_{ij} | \sigma)$ ..

consider $X_{1,2}$:

$$E(X_{1,2}) = \sum_{\sigma \in S} p(\sigma) \cdot X_{12}(\sigma) \quad \rightarrow \text{since } X_{1,2} \text{ zero if not invesion}$$

$$= \sum_{(i,j) \text{inv.}} \frac{1}{n!} = \frac{1}{n!} \cdot (\text{\# of permutation } \sigma \text{ such that } i,j \text{ is inverted.}$$

$$= \sum \frac{1}{n!} \frac{n!}{2} = \frac{1}{2}$$

recognize that for every inverted pair of #, there is non-inverted pair.

= So 50 chance of inversion for ij pair.

Now,

$$E(X) = E(\sum_{i<j} X_{ij}) = \sum_{1 \le i \le j \le n} E(X_{ij}) = \sum_{1 \le i < j \le n} (\tfrac{1}{2})$$

$$= n(\tfrac{n-1}{2})(\tfrac{1}{2})$$

$$= \frac{n(n-1)}{4} \quad ?$$

Def A p-Bernoulli trial is the flipping of a 0/1 coin for which $p(1) = p$ and $p(0) = 1-p$.

Proposed Experiment: Flip a coin until we see "1".

Q: How many times should you expect to flip?

A: S = {all sequences of the form $(1), (0,1), (0,0,1) \ldots (0, \ldots, 0, 1) \ldots$}

and $|S| = \infty$.

what does $p(n+1)$ mean?

= find the prob of n zeros followed by 1

$= p(\underbrace{0, \dots, 0}_{n}, 1)$

Assuming independence of flipping coins,

$p(\text{first flip} = 0) = 1 - p := q$

$p(\text{first flip} = 0) = q^2$

$p(1,2,3^{rd} \text{ flip} = 0) = q^3 \quad \dots \quad p(\text{first } n \text{ flip} = 0) = q^n$

$\Rightarrow p(n+1) = q^n \cdot p$

and $S = \{s_1 = 1, \; s_2 = (0,1), \; s^3 = (0,0,1), \; \dots \; s_{n+1} = \{0, \dots, 0, 1\}$

$\qquad p \qquad q \cdot p \qquad q^2 \cdot p \qquad q^n p.$

A sample space of this type of prob is called geometric distribution.

To calculate average number of flips,

→ random variable X

$X(s_{flips}) = k$ (simply the length of sequence ... # of flipps needed)

So $E(X) = \sum_{s \in S} p(s \cdot X(s))$

$= \sum_{n \in \mathbb{N}}^{\infty} (q^{n-1})(p)\, n = \overset{\text{constant}}{p} \sum_{n=1}^{\infty} q^{n-1} \cdot n$ familar? = derivative

starts w/ one

$= p \sum_{n=1}^{\infty} (q^n)'$

nice thing about derivative? Can make this statement

$q + q^2 + q^3 + \dots q^n$

↳ similar to $1 + q + q^2 + q^3 \dots$

~~thus~~; which is equal to $\frac{1}{1-q}$ if $|q| < 1$.

$= p\left(\sum_{n=1}^{\infty} q^n\right)'$

$= p\left(\frac{1}{1-q} - 1\right)'$

quotient rule

$= p\left(\frac{1}{(1-q)^2}\right)$

$= p\left(\frac{1}{p^2}\right) = \frac{1}{p}$

reasonable answer?

if $p = 0$ (never get 1) $E(x) = \frac{1}{0}$: infinite rolls ✓

if $p = 1$ (always get 1) $E(x) = 1$: 1 roll ✓

Recall: $p(E) \cdot p(F) = p(E \cap F)$

if E and F are independent

Def: 2 random variables are independent if for each pair of $\alpha, \beta \in \mathbb{R}$ and possible values mapped by x, y.

$$p(\{x(s) = \alpha\}) \cdot p(\{y(s) = \beta\}) = p(\{x(s) = \alpha, y(s) = \beta\})$$

subcollection of S

Ex $S = \{1, 2, 3, 4, 5, 6\}$

$X(s) = s$

$Y(s) = s^2$

Note: Since knowing $X(s)$ reveals all info about $Y(s)$, we do not expect these two variables to be independent

Q: Are X & Y indp?

consider $\alpha = \{1, 2, 3, 4, 5, 6\}$ and $\beta = \{1, 4, 9, 16, 25, 36\}$

$= 30$ checks need be done.

1. $\alpha = 1 \quad \beta = 1$

$p(X=1) = \frac{1}{6}$

$p(Y=1) = \frac{1}{6}$

$p(X=1, Y=1) = \frac{1}{6}$

since $\frac{1}{6} \cdot \frac{1}{6} \neq \frac{1}{6}$ $X + Y$ are dependent.

Ex S = two dice

$$= \{\underbrace{(1,1), (2,2), (3,3) \ldots, (6,6)}_{p = \frac{1}{36}} \; \underbrace{(1,2)\ (1,3) \ldots (5,6)}_{p = \frac{1}{18}}\}$$

$X((i,j)) = i + j$

$Y((i,j)) = \begin{cases} 0 & \text{if } i \cdot j \text{ even} \\ 1 & \text{if } i \cdot j \text{ odd} \end{cases}$

Q: X, Y indp?

X has values $2, \ldots, 12$

Y has values 0 & 1

] 24 comb to check for indp.

Check 1 $\alpha = 2 \quad \beta = 0$

$p(X=2) = \frac{1}{36}$ (only $(1,1)$)

$p(Y=0) = 1 -$ prob that both i, j are odd $= 1-(0.5)(0.5) = 0.75$

$P(X=2, Y=0) = 0$ (can't be possible)

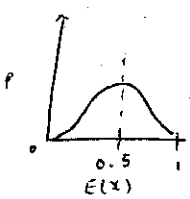
X, Y are not indpendent.

We now look at <u>variance</u>;

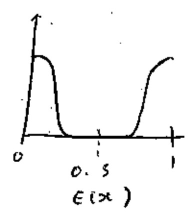
2 random variables w/ equal expected value can behave very diff.

eg. $S = [0, 1]$

one could have



versus



$\therefore$ $E(X)$ is certainly not an exhaustive knowledge.

<u>Variance</u> is a second degree approximation: it measures the average distance to the average.

<u>Def</u>: $V(X) = \sum (E(X) - X(s))^{2}$ — to remove $-, +$ distinction.

$\therefore v(x) \geq 0$

<u>FACT</u>: $V(X) = E(X^2) - [E(X)]^2$ (quite magical?)

e.g. variance of geometric distribution.

$S = \{1, 2, \ldots, \ldots\}$ $\quad p(n+1) = q^n \cdot p$

$X(i) = i$ # of times we flip until 1

$E(X) = \frac{1}{p}$ (established previously)

$V(X) =$ how closely does X stick to $E(X)$?

<u>Thm</u>: If two variables are indp, variance becomes additive:

$V(X) + V(Y) = V(X+Y)$

In perticular, we consider

V(a sequence of flips) $= \sum V$(seperate flips)