This is a 2 hour practice exam. Do 8 of the 11.

1. Let $f : X \to Y$ with X, Y both metric spaces. Prove that if for all $x, x_n \in X, x_n \to x$ we have $f(x_n) \to f(x)$ then f is continuous at x.

2. Let
$$f = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$$
Find (with proof) the set on which f is continuous.

3. Proof or counter-example: the product of 2 uniformly continuous functions is uniformly continuous.

4. Find **without proof** a function who has a derivative (everywhere), but the derivative is discontinuous.

5. Converge absolutely? (with proof)
$$\sum_{n=100}^{\infty} \frac{\log^2(n^n)\sin(n^2+4n!)}{(n^2-4n+7)^2}$$

6. For what x does the following sum converge?
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+x^n}$$

7. $f : \mathbb{R} \to \mathbb{R}$, continuous and
$$\lim_{|x|\to\infty} f(x) = 0 = \lim_{|x|\to\infty} f(1/x)$$
Show $f(\mathbb{R})$ is compact.

8. $f : \mathbb{R} \to \mathbb{R}, f(0) = 0$. For all $|x_n| \to \infty, \sum x_n f(1/x_n)$ converges. Show $f'(0)$ exists and is zero.

9. Consider $\{x^n\}_{n=1}^{\infty}$ with domain $[0, 1]$ and show this family is not equicontinuous.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous, and show there is some function of the form $y = m|x| + b$ such that $y \geq f \geq -y$.

11. Let $f_{n,m}(x) = e^{-m}x^n$ on $[0, 1/2]$ and set $F_m = \sum_{n=0}^{\infty} f_{n,m}$.

    (a) Show F_m converges absolutely and uniformly.

    (b) Show $\{F_m\}$ has a uniformly convergent subsequence. (You may use the fact that $\sum_{n=1}^{\infty} n(1/2)^{n-1}$ converges without proof).