(Review)

Reconstructed signal $\qquad$ $x_a(t)$: original signal.

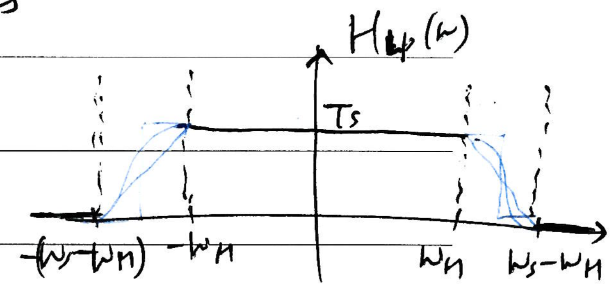
$$x_r(t) = \underline{x_s(t)} * h_{LP}(t)$$

$$= \left\{ \sum_n x_a(nT_s)\,\delta(t-nT_s) \right\} * h_{LP}(t)$$

$$= \sum_n x_a(nT_s)\, h_{LP}(t-nT_s)$$

* Assuming $\omega_s > 2\omega_M$ and $H_{LP}(\omega)$ satisfies

$$H_{LP}(\omega) = \begin{cases} T_s, & |\omega| < \omega_M \\ 0, & |\omega| > \omega_s - \omega_M \end{cases}$$



then, $\underline{x_r(t) = x_a(t)}$ ⇒ "Perfect Reconstruction"

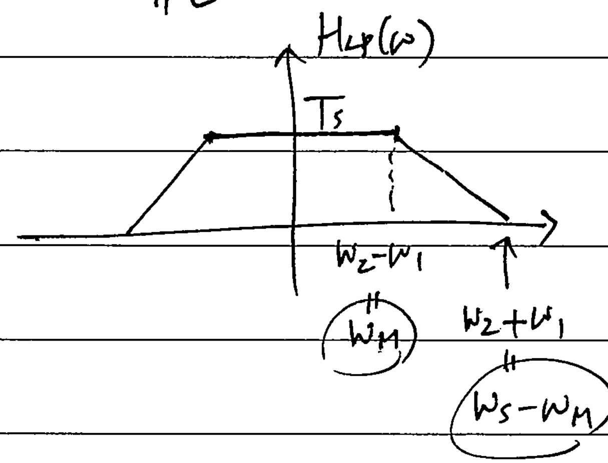
* So far, we've used $h_{LP}(t) = \dfrac{\sin\left(\frac{\omega_s}{2}t\right)}{\frac{\omega_s}{2}t}$ $\quad \left(\omega_s = 2\pi f_s = \dfrac{2\pi}{T_s}\right)$

($\omega_s > 2\omega_M$)

- With oversampling, there are many other possibilities for the interpolating LPF, $h_{LP}(t)$, that will yield perfect reconstruction $x_r(t) = x_a(t)$

- Example : $h_{LP}(t) = T_s \dfrac{\sin(\omega_c t)}{\pi t}$, $\omega_M < \omega_c < \omega_s - \omega_M$

- Example : $h_{LP}(t) = \dfrac{\pi}{\omega_1} T_s \dfrac{\sin(\omega_1 t)}{\pi t} \dfrac{\sin(\omega_2 t)}{\pi t}$



require (to avoid aliasing) $\begin{cases} \omega_2 - \omega_1 = \omega_M \\ \omega_2 + \omega_1 = \omega_s - \omega_M \end{cases}$

$2\omega_2 = \omega_s \rightarrow \omega_2 = \dfrac{\omega_s}{2}$

$\omega_1 = \omega_2 - \omega_M \rightarrow \omega_1 = \dfrac{\omega_s}{2} - \omega_M$

"With Oversampling"

$\hookrightarrow$ $x_a(t) = \sum_n x_a(nT_s)\, h_{LP}(t - nT_s)$

$h_{LP}(t) = \dfrac{\pi}{\frac{\omega_s}{2} - \omega_M} \cdot \dfrac{\sin\left(\frac{\omega_s}{2} t\right)}{\frac{\omega_s}{2} t} \cdot \dfrac{\sin\left(\left(\frac{\omega_s}{2} - \omega_M\right) t\right)}{\pi t}$

Review & Further Insights Into Fourier Transform

$$\begin{cases} FT: & X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ IFT: & x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \end{cases}$$

For $x(t)$ periodic with period T

Fourier series : $x(t) = \sum_k a_k e^{jk\frac{2\pi}{T}t}$

(sum of sinewaves)

- Examine IFT integral
  - *remember Integration = "area under curve"
  - → discretize I-FT integral = "area under rectangles"

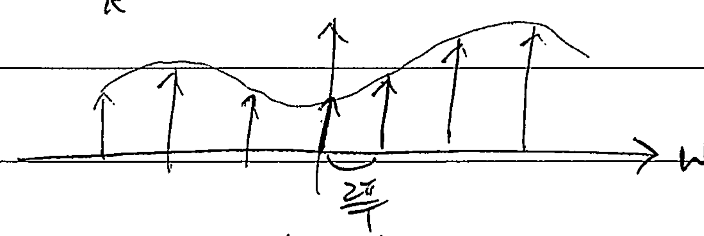
$$x(t) \approx \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\Delta\omega) e^{jk\Delta\omega t} \underbrace{\Delta\omega}_{d\omega}$$

$$\approx \sum_k \frac{\Delta\omega}{2\pi} X(k\Delta\omega) e^{jk\Delta\omega t} \quad \text{(sum of sinewaves)}$$

- When we let $\Delta\omega \to 0$ to get back to IFT Integral, we see that for
  - → aperiodic $x(t)$ is an infinite sum of sinewaves infinitesimally close in freq.
  - → the energy of an aperiodic signal is spread over a continuum of freq.
  - → in contrast to periodic signal, for which energy is at equi-spaced discrete freq.

(1) periodic $x(t)$: infinite energy, finite power

$$x(t) = \sum_k a_k e^{jk\frac{2\pi}{T}t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \sum_k a_k \delta\left(\omega - k\frac{2\pi}{T}\right)$$

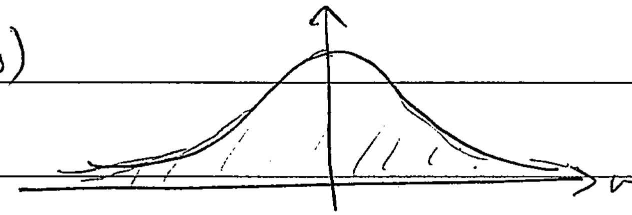


$$\text{avg power} = \frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 \, dt$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2$$: sum of powers in all the sinewaves at the harmonic freq.

(2) aperiodic $x(t)$: finite energy

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$



$$\text{Energy} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

energy spread over a continuum of freq.

※ Review FT : $\omega \longleftrightarrow f$ (← Hz) $(\omega = 2\pi f)$

FT : $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, dt$

IFT : $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \, df$

If $x(t) \xleftrightarrow{\mathcal{F}} X(f)$

- Duality : $X(t) \xleftrightarrow{\mathcal{F}} x(-f)$
- Multiplication in Time : $x(t)\,y(t) \xleftrightarrow{\mathcal{F}} X(f) * Y(f)$
- Sinewave : $e^{j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \delta(f-f_0)$
- ~~Energy~~ : $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$

  (Parseval's Theorem)
- Gaussian pulse $x(t) = e^{-\pi t^2} \xleftrightarrow{\mathcal{F}} X(f) = e^{-\pi f^2}$

  $(\sigma = \frac{1}{\sqrt{2\pi}})$
- general Gaussian $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{\mathcal{F}} e^{-\frac{1}{2}(2\pi\sigma)^2 f^2}$
- Train of delta $\sum_n \delta(t-n) \xleftrightarrow{\mathcal{F}} \sum_k \delta(f-k)$

  $\sum_n \delta(t-nT_0) \xleftrightarrow{\mathcal{F}} \sum_k \frac{1}{T_0}\delta(f-\frac{k}{T_0})$