Instructions:

1. Wait for the “BEGIN” signal before opening this booklet. In the meantime, read the instructions below and fill out the requested info.

2. **When the end of the exam is announced, you must stop writing immediately.** Anyone caught writing after the exam is over will get a grade of zero.

3. At the end of this document is a 5 page table of formulas and 4 pages of scratch paper. You may detach these *once the exam begins*. Each formula is labeled with a number. To save time, you may use these numbers to specify which formula you are using. If you use a non-trivial formula which is *not* contained in this table, you must explain why it is true in order to get full credit. (For example, Euler’s formula and the geometric series formula are trivial; the Fourier transform of a function is not.)

4. This is a closed book exam. The only personal items allowed are pens/pencils, erasers and something to drink. Anything else is strictly forbidden.

5. You must keep your eyes on your exam at all times. Looking around is strictly forbidden.

Name: ____________________________

Email: ____________________________

Signature: _________________________

<table>
<thead>
<tr>
<th>Itemized Scores</th>
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</thead>
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<tr>
<td>Problem 1:</td>
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<td>Problem 2:</td>
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<td>Problem 3:</td>
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<tr>
<td>Problem 4:</td>
</tr>
<tr>
<td>Problem 5:</td>
</tr>
</tbody>
</table>

1
(10 pts) 1. State the sampling theorem. (You may use your own words but be precise!)
(15 pts) 2. Obtain the inverse Laplace transform of

\[ X(s) = \frac{1}{s^2 + 3s + 2}, \text{ ROC: } -2 < \text{Re}(s) < -1. \]
(20 pts) 3. Define what is an LTI system by filling in the blanks and the diagrams below.

- L stands for \[\text{Linear},\] which means that Diagram 1 and Diagram 2 below produce the same output:

  \[
  x_1(t) \rightarrow \hspace{2cm} \rightarrow y(t) \\
  x_2(t) \rightarrow \\
  \]

  \[
  x_1(t) \rightarrow \hspace{2cm} \rightarrow y(t) \\
  x_2(t) \rightarrow \\
  \]

- TI stands for \[\text{Time Invariant},\] which means that Diagram 3 and Diagram 4 below produce the same output:

  \[
  x(t) \rightarrow \hspace{2cm} \rightarrow y(t) \\
  \]

  \[
  x(t) \rightarrow \hspace{2cm} \rightarrow y(t) \\
  \]
(20 pts) 4. Which of these signals cannot possibly be the response of an LTI system to the input \(x[n] = (1 - j)^n\). (No justification needed. Simply circle the signals.)

1. \(y[n] = u[n]\)
2. \(y[n] = (1 - j)^n\)
3. \(y[n] = n(1 - j)^n\)
4. \(y[n] = u[n](1 - j)^n\)
5. \(y[n] = j(1 - j)^n\)
6. \(y[n] = 1^n + (-j)^n\)
7. \(y[n] = (1 - j)^{2n}\)
8. \(y[n] = (1 - 2j)^n\)
9. \(y[n] = \sqrt{2}e^{-j\pi/4}\)
10. \(y[n] = e^{-j\pi/4}\)

You may use the space below as scratch. (This work will not be graded.)
(15 pts) 5. An LTI system has unit impulse response \( h[n] = u[n - 2] \). Use convolution to compute the system’s response to the input \( x[n] = (\frac{1}{2})^{n+2} u[n + 2] \). (Simplify your answer until all \( \sum \) signs disappear.)
(5pts) 6. Is the DT signal \( x[n] = \cos(7\pi n) \) periodic? (Answer yes/no and justify your answer.)

(5 pts) 7. The input \( x(t) \) and output \( y(t) \) of a system are related by the equation 

\[
y(t) = x(t) + x(t - 1) + x(t + 1).
\]

What is the unit impulse response of this system?
(15 pts) 8. Compute the Fourier series coefficients of the DT signal $x[n] = j^{n+2}$.
(Hint: the answer can be obtained in three simple steps.)
(15 pts) 9. Using the definition of the z-transform (i.e. do not simply take the answer from the table), compute the inverse z-transform of

\[ X(z) = \frac{z}{1 + \frac{1}{4}z}, |z| < 4. \]
Table

DT Signal Energy and Power

\[
E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2 
\]

\[
P_\infty = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} |x[n]|^2 
\]

CT Signal Energy and Power

\[
E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt 
\]

\[
P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt 
\]

Fourier Series of CT Periodic Signals with period \( T \)

\[
x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \left( \frac{2\pi}{T} \right) t} 
\]

\[
a_k = \frac{1}{T} \int_{0}^{T} x(t) e^{-j k \left( \frac{2\pi}{T} \right) t} dt 
\]

Fourier Series of DT Periodic Signals with period \( N \)

\[
x[n] = \sum_{k=0}^{N-1} a_k e^{j k \left( \frac{2\pi}{N} \right) n} 
\]

\[
a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j k \left( \frac{2\pi}{N} \right) n} 
\]
CT Fourier Transform

$$\text{F.T. : } X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$  \hspace{1cm} (9)

$$\text{Inverse F.T.: } x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$  \hspace{1cm} (10)

Properties of CT Fourier Transform

Let \( x(t) \) be a continuous-time signal and denote by \( X(\omega) \) its Fourier transform. Let \( y(t) \) be another continuous-time signal and denote by \( Y(\omega) \) its Fourier transform.

<table>
<thead>
<tr>
<th>Signal</th>
<th>FT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity: ( ax(t) + by(t) )</td>
<td>( aX(\omega) + bY(\omega) )</td>
</tr>
<tr>
<td>Time Shifting: ( x(t - t_0) )</td>
<td>( e^{-j\omega t_0} X(\omega) )</td>
</tr>
<tr>
<td>Frequency Shifting: ( e^{j\omega t_0} x(t) )</td>
<td>( X(\omega - \omega_0) )</td>
</tr>
<tr>
<td>Time and Frequency Scaling: ( x(at) )</td>
<td>( \frac{1}{</td>
</tr>
<tr>
<td>Multiplication: ( x(t)y(t) )</td>
<td>( \frac{1}{2\pi} X(\omega) \ast Y(\omega) )</td>
</tr>
<tr>
<td>Convolution: ( x(t) \ast y(t) )</td>
<td>( X(\omega)Y(\omega) )</td>
</tr>
<tr>
<td>Differentiation in Time: ( \frac{d}{dt} x(t) )</td>
<td>( j\omega X(\omega) )</td>
</tr>
</tbody>
</table>

Some CT Fourier Transform Pairs

$$e^{j\omega t} \xrightarrow{\mathcal{F}} 2\pi \delta(\omega - \omega_0)$$  \hspace{1cm} (18)

$$1 \xrightarrow{\mathcal{F}} 2\pi \delta(\omega)$$  \hspace{1cm} (19)

$$\frac{\sin W t}{\pi t} \xrightarrow{\mathcal{F}} u(\omega + W) - u(\omega - W)$$  \hspace{1cm} (20)

$$u(t + T_1) - u(t - T_1) \xrightarrow{\mathcal{F}} \frac{2\sin(\omega T_1)}{\omega}$$  \hspace{1cm} (21)

$$\delta(t) \xrightarrow{\mathcal{F}} 1$$  \hspace{1cm} (22)

$$e^{-at}u(t), \Re\{a\} > 0 \xrightarrow{\mathcal{F}} \frac{1}{a + j\omega}$$  \hspace{1cm} (23)

$$te^{-at}u(t), \Re\{a\} > 0 \xrightarrow{\mathcal{F}} \frac{1}{(a + j\omega)^2}$$  \hspace{1cm} (24)
**DT Fourier Transform**

Let $x[n]$ be a discrete-time signal and denote by $X(\omega)$ its Fourier transform.

\[
F.T.: X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} \quad (25)
\]

Inverse F.T.: $x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega)e^{jn\omega} d\omega \quad (26)$

**Properties of DT Fourier Transform**

Let $x(t)$ be a signal and denote by $X(\omega)$ its Fourier transform. Let $y(t)$ be another signal and denote by $Y(\omega)$ its Fourier transform.

**Signal F.T.**

- **Linearity:** $ax[n] + by[n] \rightarrow aX(\omega) + bY(\omega) \quad (27)$
- **Time Shifting:** $x[n - n_0] \rightarrow e^{-jn_0\omega}X(\omega) \quad (28)$
- **Frequency Shifting:** $e^{j\omega_0 n}x[n] \rightarrow X(\omega - \omega_0) \quad (29)$
- **Time Reversal:** $x[-n] \rightarrow X(-\omega) \quad (30)$
- **Multiplication:** $x[n]y[n] \rightarrow \frac{1}{2\pi} X(\omega) \ast Y(\omega) \quad (31)$
- **Convolution:** $x[n] * y[n] \rightarrow X(\omega)Y(\omega) \quad (32)$
- **Differencing in Time:** $x[n] - x[n - 1] \rightarrow (1 - e^{-jn\omega})X(\omega) \quad (33)$

**Some DT Fourier Transform Pairs**

\[
\sum_{k=0}^{N-1} a_k e^{j\frac{2\pi k n}{N}} \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi k}{N}) \quad (34)
\]

\[
e^{j\omega_0 n} \leftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l) \quad (35)
\]

\[
1 \leftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) \quad (36)
\]

\[
\frac{\sin Wn}{\pi n}, \ 0 < W < \pi \ \rightarrow \ X(\omega) = \begin{cases} 1, & 0 < |\omega| < W \\ 0, & \pi \geq |\omega| > W \end{cases} \quad (37)
\]

\[
\delta[n] \rightarrow 1 \quad (38)
\]

\[
u[n] = \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \quad (39)
\]

\[
a^n u[n], |a| < 1 \ \rightarrow \ \frac{1}{1 - ae^{-j\omega}} \quad (40)
\]

\[(n + 1)a^n u[n], |a| < 1 \ \rightarrow \ \frac{1}{(1 - ae^{-j\omega})^2} \quad (41)
\]
Laplace Transform

\[ X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \] (42)

Properties of Laplace Transform

Let \( x(t), x_1(t) \) and \( x_2(t) \) be three CT signals and denote by \( X(s), X_1(s) \) and \( X_2(s) \) their respective Laplace transform. Let \( R \) be the ROC of \( X(s) \), let \( R_1 \) be the ROC of \( X_1(z) \) and let \( R_2 \) be the ROC of \( X_2(s) \).

<table>
<thead>
<tr>
<th>Property</th>
<th>Signal</th>
<th>L.T.</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>( ax_1(t) + bx_2(t) )</td>
<td>( aX_1(s) + bX_2(s) )</td>
<td>At least ( R_1 \cap R_2 )</td>
</tr>
<tr>
<td>Time Shifting</td>
<td>( x(t - t_0) )</td>
<td>( e^{-st_0}X(s) )</td>
<td>( R )</td>
</tr>
<tr>
<td>Shifting in s</td>
<td>( e^{st}x(t) )</td>
<td>( X(s - s_0) )</td>
<td>( R + s_0 )</td>
</tr>
<tr>
<td>Conjugation</td>
<td>( x^*(t) )</td>
<td>( X^<em>(s^</em>) )</td>
<td>( R )</td>
</tr>
<tr>
<td>Time Scaling</td>
<td>( x(at) )</td>
<td>( \frac{1}{</td>
<td>a</td>
</tr>
<tr>
<td>Convolution</td>
<td>( x_1(t) \ast x_2(t) )</td>
<td>( X_1(s)X_2(s) )</td>
<td>At least ( R_1 \cap R_2 )</td>
</tr>
<tr>
<td>Differentiation in Time</td>
<td>( \frac{d}{dt}x(t) )</td>
<td>( sX(s) )</td>
<td>At least ( R )</td>
</tr>
<tr>
<td>Differentiation in s</td>
<td>( tx(t) )</td>
<td>( \frac{dX(s)}{ds} )</td>
<td>( R )</td>
</tr>
<tr>
<td>Integration</td>
<td>( \int_{-\infty}^{t} x(\tau)d\tau )</td>
<td>( \frac{1}{s}X(s) )</td>
<td>At least ( R \cap \Re{s} \geq 0 )</td>
</tr>
</tbody>
</table>

Some Laplace Transform Pairs

<table>
<thead>
<tr>
<th>Signal</th>
<th>LT</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(t) )</td>
<td>1</td>
<td>all ( s )</td>
</tr>
<tr>
<td>( e^{-\alpha t}u(t) )</td>
<td>( \frac{1}{s + \alpha} )</td>
<td>( \Re{s} &gt; -\alpha )</td>
</tr>
<tr>
<td>( -e^{-\alpha t}u(-t) )</td>
<td>( \frac{1}{s + \alpha} )</td>
<td>( \Re{s} &lt; -\alpha )</td>
</tr>
</tbody>
</table>
$z$-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad (55)$$

Properties of $z$-Transform

Let $x[n]$, $x_1[n]$ and $x_2[n]$ be three DT signals and denote by $X(z)$, $X_1(z)$ and $X_2(z)$ their respective $z$-transform. Let $R$ be the ROC of $X(z)$, let $R_1$ be the ROC of $X_1(z)$ and let $R_2$ be the ROC of $X_2(z)$.

<table>
<thead>
<tr>
<th>Signal</th>
<th>$z$-T.</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity: $ax_1[n] + bx_2[n]$</td>
<td>$aX_1(z) + bX_2(z)$</td>
<td>At least $R_1 \cap R_2$</td>
</tr>
<tr>
<td>Time Shifting: $x[n - n_0]$</td>
<td>$z^{-n_0}X(z)$</td>
<td>$R$, but perhaps adding/deleting $z = 0$</td>
</tr>
<tr>
<td>Time Shifting: $x[-n]$</td>
<td>$X(z^{-1})$</td>
<td>$R^{-1}$</td>
</tr>
<tr>
<td>Scaling in $z$: $e^{j\omega_0}x[n]$</td>
<td>$X(e^{-j\omega_0}z)$</td>
<td>$R$</td>
</tr>
<tr>
<td>Conjugation: $x^*[n]$</td>
<td>$X^<em>(z^</em>)$</td>
<td>$R$</td>
</tr>
<tr>
<td>Convolution: $x_1[n] * x_2[n]$</td>
<td>$X_1(z)X_2(z)$</td>
<td>At least $R_1 \cap R_2$</td>
</tr>
</tbody>
</table>

Some $z$-Transform Pairs

<table>
<thead>
<tr>
<th>Signal</th>
<th>LT</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u[n]$</td>
<td>$\frac{1}{1 - z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$-u[-n - 1]$</td>
<td>$\frac{1}{1 - z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$\alpha^n u[n]$</td>
<td>$\frac{1}{1 - \alpha z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$-\alpha^n u[-n - 1]$</td>
<td>$\frac{1}{1 - \alpha z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$\delta[n]$</td>
<td>$1$</td>
<td>all $z$</td>
</tr>
</tbody>
</table>
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