ECE 301

Division 3, Fall 2007

Instructor: Mimi Boutin

Final Examination

Instructions:

1. Wait for the "BEGIN" signal before opening this booklet. In the meantime, read the instructions below and fill out the requested info.
2. **When the end of the exam is announced, you must stop writing immediately**. Anyone caught writing after the exam is over will get a grade of zero.
3. At the end of this document is a 5 page table of formulas and 4 pages of scratch paper. You may detach these **once the exam begins**. Each formula is labeled with a number. To save time, you may use these numbers to specify which formula you are using. If you use a non-trivial formula which is *not* contained in this table, you must explain why it is true in order to get full credit. (For example, Euler's formula and the geometric series formula are trivial; the Fourier transform of a function is not.)
4. This is a closed book exam. The only personal items allowed are pens/pencils, erasers and something to drink. Anything else is strictly forbidden.
5. You must keep your eyes on your exam at all times. Looking around is strictly forbidden.

Name: ______________________________

Email: ______________________________

Signature: ______________________________

Itemized Scores

Problem 1:	Problem 6:
Problem 2:	Problem 7:
Problem 3:	Problem 8:
Problem 4:	Problem 9:
Problem 5:	
Total:	

(10 pts) **1.** State the sampling theorem. (You may use your own words but be precise!)

(15 pts) **2.** Obtain the inverse Laplace transform of

$$X(s) = \frac{1}{s^2 + 3s + 2}, \text{ROC:} -2 < \text{Re}(s) < -1.$$

(20 pts) **3.** Define what is an LTI system by filling in the blanks and the diagrams below.

- L stands for ______________________,

  which means that Diagram 1 and Diagram 2 below produce the same output:

Diagram 1

$x_1(t) \rightarrow$

$\rightarrow y(t)$

$x_2(t) \rightarrow$

Diagram 2

$x_1(t) \rightarrow$

$\rightarrow y(t)$

$x_2(t) \rightarrow$

- TI stands for ______________________,

  which means that Diagram 3 and Diagram 4 below produce the same output:

Diagram 3

$x(t) \rightarrow$ $\rightarrow y(t)$

Diagram 4

$x(t) \rightarrow$ $\rightarrow y(t)$

(20 pts) **4.** Which of these signals cannot possibly be the response of an LTI system to the input $x[n] = (1-j)^n$. (No justification needed. Simply circle the signals.)

1. $y[n] = u[n]$
2. $y[n] = (1-j)^n$
3. $y[n] = n(1-j)^n$
4. $y[n] = u[n](1-j)^n$
5. $y[n] = j(1-j)^n$
6. $y[n] = 1^n + (-j)^n$
7. $y[n] = (1-j)^{2n}$
8. $y[n] = (1-2j)^n$
9. $y[n] = \sqrt{2}e^{-j\frac{\pi}{4}}$
10. $y[n] = e^{-j\frac{\pi}{4}}$

You may use the space below as scratch. (This work will *not* be graded.)

(15 pts) **5.** An LTI system has unit impulse response $h[n] = u[n-2]$. Use convolution to compute the system's response to the input $x[n] = \left(\frac{1}{2}\right)^{n+2} u[n+2]$. (Simplify your answer until all $\sum$ signs disappear.)

(5pts) **6.** Is the DT signal $x[n] = \cos(7\pi n)$ periodic? (Answer yes/no and justify your answer.)

(5 pts) **7.** The input $x(t)$ and output $y(t)$ of a system are related by the equation

$$y(t) = x(t) + x(t-1) + x(t+1).$$

What is the unit impulse response of this system?

(15 pts) **8.** Compute the Fourier series coefficients of the DT signal $x[n] = j^{n+2}$.
(Hint: the answer can be obtained in three simple steps.)

(15 pts) **9.** Using the definition of the z-transform (i.e. do not simply take the answer from the table), compute the inverse z-transform of

$$X(z) = \frac{z}{1 + \frac{1}{4}z}, |z| < 4.$$

Table

DT Signal Energy and Power

$$E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{1}$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{2}$$

CT Signal Energy and Power

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \tag{3}$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \tag{4}$$

Fourier Series of CT Periodic Signals with period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \tag{5}$$

$$a_k = \frac{1}{T}\int_0^T x(t) e^{-jk\left(\frac{2\pi}{T}\right)t} dt \tag{6}$$

Fourier Series of DT Periodic Signals with period N

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \tag{7}$$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} \tag{8}$$

CT Fourier Transform

$$\text{F.T. : } \mathcal{X}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad (9)$$

$$\text{Inverse F.T.: } x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{X}(\omega)e^{j\omega t}d\omega \quad (10)$$

Properties of CT Fourier Transform

Let $x(t)$ be a continuous-time signal and denote by $\mathcal{X}(\omega)$ its Fourier transform. Let $y(t)$ be another continuous-time signal and denote by $\mathcal{Y}(\omega)$ its Fourier transform.

	Signal	*FT*	
Linearity:	$ax(t) + by(t)$	$a\mathcal{X}(\omega) + b\mathcal{Y}(\omega)$	(11)
Time Shifting:	$x(t - t_0)$	$e^{-j\omega t_0}\mathcal{X}(\omega)$	(12)
Frequency Shifting:	$e^{j\omega_0 t}x(t)$	$\mathcal{X}(\omega - \omega_0)$	(13)
Time and Frequency Scaling:	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{\omega}{a}\right)$	(14)
Multiplication:	$x(t)y(t)$	$\frac{1}{2\pi}\mathcal{X}(\omega) * \mathcal{Y}(\omega)$	(15)
Convolution:	$x(t) * y(t)$	$\mathcal{X}(\omega)\mathcal{Y}(\omega)$	(16)
Differentiation in Time:	$\frac{d}{dt}x(t)$	$j\omega\mathcal{X}(\omega)$	(17)

Some CT Fourier Transform Pairs

$$e^{j\omega_0 t} \xrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0) \quad (18)$$

$$1 \xrightarrow{\mathcal{F}} 2\pi\delta(\omega) \quad (19)$$

$$\frac{\sin Wt}{\pi t} \xrightarrow{\mathcal{F}} u(\omega + W) - u(\omega - W) \quad (20)$$

$$u(t + T_1) - u(t - T_1) \xrightarrow{\mathcal{F}} \frac{2\sin(\omega T_1)}{\omega} \quad (21)$$

$$\delta(t) \xrightarrow{\mathcal{F}} 1 \quad (22)$$

$$e^{-at}u(t), \mathcal{R}e\{a\} > 0 \xrightarrow{\mathcal{F}} \frac{1}{a + j\omega} \quad (23)$$

$$te^{-at}u(t), \mathcal{R}e\{a\} > 0 \xrightarrow{\mathcal{F}} \frac{1}{(a + j\omega)^2} \quad (24)$$

DT Fourier Transform

Let $x[n]$ be a discrete-time signal and denote by $X(\omega)$ its Fourier transform.

$$\text{F.T.:}\mathcal{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (25)$$

$$\text{Inverse F.T.: } x[n] = \frac{1}{2\pi}\int_{2\pi} \mathcal{X}(\omega)e^{j\omega n}d\omega \quad (26)$$

Properties of DT Fourier Transform

Let $x(t)$ be a signal and denote by $\mathcal{X}(\omega)$ its Fourier transform. Let $y(t)$ be another signal and denote by $\mathcal{Y}(\omega)$ its Fourier transform.

	Signal	*F.T.*	
Linearity:	$ax[n] + by[n]$	$a\mathcal{X}(\omega) + b\mathcal{Y}(\omega)$	(27)
Time Shifting:	$x[n - n_0]$	$e^{-j\omega n_0}\mathcal{X}(\omega)$	(28)
Frequency Shifting:	$e^{j\omega_0 n}x[n]$	$\mathcal{X}(\omega - \omega_0)$	(29)
Time Reversal:	$x[-n]$	$\mathcal{X}(-\omega)$	(30)
Multiplication:	$x[n]y[n]$	$\frac{1}{2\pi}\mathcal{X}(\omega) * \mathcal{Y}(\omega)$	(31)
Convolution:	$x[n] * y[n]$	$\mathcal{X}(\omega)\mathcal{Y}(\omega)$	(32)
Differencing in Time:	$x[n] - x[n-1]$	$(1 - e^{-j\omega})\mathcal{X}(\omega)$	(33)

Some DT Fourier Transform Pairs

$$\sum_{k=0}^{N-1} a_k e^{jk(\frac{2\pi}{N})n} \xrightarrow{\mathcal{F}} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi k}{N}) \quad (34)$$

$$e^{j\omega_0 n} \xrightarrow{\mathcal{F}} 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l) \quad (35)$$

$$1 \xrightarrow{\mathcal{F}} 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) \quad (36)$$

$$\frac{\sin Wn}{\pi n}, 0 < W < \pi \xrightarrow{\mathcal{F}} \mathcal{X}(\omega) = \begin{cases} 1, & 0 \leq |\omega| < W \\ 0, & \pi \geq |\omega| > W \end{cases} \quad (37)$$

$\mathcal{X}(\omega)$periodic with period 2π

$$\delta[n] \xrightarrow{\mathcal{F}} 1 \quad (38)$$

$$u[n] \xrightarrow{\mathcal{F}} \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \quad (39)$$

$$\alpha^n u[n], |\alpha| < 1 \xrightarrow{\mathcal{F}} \frac{1}{1 - \alpha e^{-j\omega}} \quad (40)$$

$$(n+1)\alpha^n u[n], |\alpha| < 1 \xrightarrow{\mathcal{F}} \frac{1}{(1 - \alpha e^{-j\omega})^2} \quad (41)$$

Laplace Transform

$$X(s) \quad = \quad \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{42}$$

Properties of Laplace Transform

Let $x(t)$, $x_1(t)$ and $x_2(t)$ be three CT signals and denote by $X(s)$, $X_1(s)$ and $X_2(s)$ their respective Laplace transform. Let R be the ROC of $X(s)$, let R_1 be the ROC of $X_1(z)$ and let R_2 be the ROC of $X_2(s)$.

	Signal	L.T.	ROC	
Linearity:	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$	(43)
Time Shifting:	$x(t - t_0)$	$e^{-st_0}X(s)$	R	(44)
Shifting in s:	$e^{s_0 t}x(t)$	$X(s - s_0)$	$R + s_0$	(45)
Conjugation:	$x^*(t)$	$X^*(s^*)$	R	(46)
Time Scaling:	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{s}{a}\right)$	aR	(47)
Convolution:	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$	(48)
Differentiation in Time:	$\frac{d}{dt}x(t)$	$sX(s)$	At least R	(49)
Differentiation in s:	$-tx(t)$	$\frac{dX(s)}{ds}$	R	(50)
Integration :	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \mathcal{R}e\{s\} > 0$	(51)

Some Laplace Transform Pairs

Signal	*LT*	*ROC*	
$\delta(t)$	1	all s	(52)
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} > -\alpha$	(53)
$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{R}e\{s\} < -\alpha$	(54)

z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad (55)$$

Properties of z-Transform

Let $x[n]$, $x_1[n]$ and $x_2[n]$ be three DT signals and denote by $X(z)$, $X_1(z)$ and $X_2(z)$ their respective z-transform. Let R be the ROC of $X(z)$, let R_1 be the ROC of $X_1(z)$ and let R_2 be the ROC of $X_2(z)$.

	Signal	z-T.	ROC	
Linearity:	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least $R_1 \cap R_2$	(56)
Time Shifting:	$x[n - n_0]$	$z^{-n_0}X(z)$	R, but perhaps adding/deleting $z = 0$	(57)
Time Shifting:	$x[-n]$	$X(z^{-1})$	R^{-1}	(58)
Scaling in z:	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R	(59)
Conjugation:	$x^*[n]$	$X^*(z^*)$	R	(60)
Convolution:	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least $R_1 \cap R_2$	(61)

Some z-Transform Pairs

Signal	LT	ROC	
$u[n]$	$\frac{1}{1 - z^{-1}}$	$\|z\| > 1$	(62)
$-u[-n-1]$	$\frac{1}{1 - z^{-1}}$	$\|z\| < 1$	(63)
$\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$\|z\| > \alpha$	(64)
$-\alpha^n u[-n-1]$	$\frac{1}{1 - \alpha z^{-1}}$	$\|z\| < \alpha$	(65)
$\delta[n]$	1	all z	(66)

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