ECE 438 Fall 2013 Homework 5 Solution

1 a). The DTFT of the reconstructed signal is:

$$X_r(w) = H_r(w)\, X_s(w)$$

where low-pass filter $H_r(w) = \begin{cases} T, & |w| < \frac{1}{2T} \\ 0, & \text{else} \end{cases}$

$X_s(w) = \text{DTFT}\{x_s(t)\}$

and $x_s(t) = \sum_{n=-\infty}^{\infty} y[n]\, \delta(t-nT)$

Then, in time domain:

$$\Rightarrow x_r(t) = h_r(t) * x_s(t)$$
$$= \text{IDTFT}\{H_r(w)\} * \sum_{n=-\infty}^{\infty} y[n]\, \delta(t-nT)$$
$$= \text{sinc}\left(\frac{t}{T}\right) * \sum_{n=-\infty}^{\infty} y[n]\, \delta(t-nT)$$
$$= \sum_{n=-\infty}^{\infty} y[n]\, \text{sinc}\left(\frac{t-nT}{T}\right)$$

b) Since $y[n] = x(nT)$

$$x_r(kT) = \sum_{n=-\infty}^{\infty} x(nT)\, \text{sinc}\left(\frac{kT-nT}{T}\right)$$
$$= \sum_{n=-\infty}^{\infty} x(nT)\, \text{sinc}(k-n)$$

because $k-n$ must be an integer, $\text{sinc}(k-n) = \begin{cases} 1, & k=n \\ 0, & \text{else} \end{cases}$

Therefore, $x_r(kT) = x(kT) \quad \forall k$.

$\Rightarrow$ this reconstruction is equal to the original signal at all sample points.

c). When Nyquist condition is met.

that is $X(w) = 0 \quad \forall w \text{ s.t. } |w| > \frac{1}{2T}$

2\. a).

$$x_r(t) = \sum_{k=-\infty}^{\infty} x(kT)\ \mathrm{rect}\left(\frac{t-\frac{T}{2}-kT}{T}\right)$$

b).

$$x_r(t) = h_0(t) * x_s(t)$$

where $x_s(t) = \mathrm{comb}_T(x(t))$

$h_0(t) = \mathrm{rect}\left(\frac{t-\frac{T}{2}}{T}\right)$

$$\Rightarrow X_r(f) = H_0(f)\, X_s(f)$$

$$= T\,\mathrm{sinc}(Tf)\, e^{-j\frac{2\pi}{2}Tf} \cdot X_s(f)$$

$$= T\,\mathrm{sinc}(Tf)\, e^{-j\frac{2\pi}{2}Tf} \cdot \frac{1}{T}\,\mathrm{rep}_{\frac{1}{T}}[X(f)]$$

c). Not band limit, but high frequency would be attenuated.