# IDEAS FOR REU'S

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Abstract. This is a growing list of problems REU students might work on with me.

## 1. Symbolic versus real powers on hypersurfaces

Required for this project is understanding of "symbolic powers" of an ideal, and a computer algebra program such as Macaulay 2.

Let $f = 0$ be a hypersurface in $\mathbb{P}^{n-1}$. (This means, $f$ is in $\mathbb{C}[x, y, z]$). By Swanson's theorem there is a $c = c(f)$ such that for all ideals $I$ in $R/(f)$ one has $I^{(cm)} \subseteq I^m$ for all $m$ and all $I$. Fixing the $I$ there is also $c(I, f)$, the smallest number such that $I^{(cm)} \subseteq I^m$ for all $m$.

Geometrically, $f$ is a hypersurface in 3-space (or a projective curve, and the ideal $I$ corresponds to a bunch of points on the curve).

Questions:

1. What is $c(f)$?
2. What is $c(I, f)$?
3. What ideals give $c(I, f) = c(f)$?
4. Compute some of them for toric surfaces, and hyperplane arrangements, when $I$ defines a reduced set of points on the curve. Relation to b-function? Milnor cohomology? lct?

Perhaps the better question to ask is about the comparison of symbolic and ordinary powers in the polynomial ring itself.

Reading material: Lazarsfeld AMS colloquium notes.

## 2. Cohen–Macaulay property and monomial ideals

For this project, you need to read what free resolutions are. (See, for example, Eisenbud's book on commutative algebra). One you know that, the following will make sense: an ideal $I$ in a polynomial ring is Cohen–Macaulay if and only if the length of its free resolution agrees with its codimension. A very special case of CM-ness arises with "complete intersections", which are those varieties that are cut out by exactly codimension many polynomials.

Monomial ideals are ideals generated by monomials. They relate to topological spaces through the very interesting "Staney–Reisner theory".

For square-free monomial ideals $I$, we know that $R/I$ is CM if and only if $I^*$ (Alexander dual) has a linear resolution. (Both CM-ness and linearity of resolutions are manifestations of good topological properties, if we want to think that way.)

Does one have a similar correspondence for non-squarefree ideals? To do this, take the topological conditions that Christine Berkesch and Laura Matusevich came up with for CM-ness. There is a notion of non-squarefree Alexander duality in Miller-Sturmfels that one can try as a replacement..

2-step Question:

1. Is there a relation between the complexes that Christine Berkesch and Laura Matusevich came up with and the multigraded Betti numbers of the Alexander dual, along the lines of

Cor 1.40 of Miller-Sturmfels? See also 1.34, 5.59 and 5.63. (There is a survey paper by Ezra Miller on various topological characterizations of CM-ness.)

2. Determine whether there is an analogue of the Eagon–Reiner theorem in the non-squarefree case. (The Eagon–Reiner theorem says that linear resolution of Alexander duals is equivalent to the original ideal being Cohen–macaulay).

Reading material: Eisenbud "Commutative algebra", Bruns–Herzog "Cohen–Macaulay rings".

## 3. Non-global rooting maps on LCM-lattices

Let $I$ be an ideal in $R = \mathbb{C}[x_1, \ldots, x_n]$. Recall that the radical of $I$ is the things in $R$ who have large powers in $I$. If $J$ is another ideal, we say that they agree up to radical if $I$ and $J$ have the same radical.

The arithmetic rank of $I$ is the smallest number of elements in $I$ that generate an ideal that agrees with $I$ up to radical. The arithmetic rank cannot be smaller than the height of $I$ by Krull's theorem. It can be bigger, though. For example, for $n = 4$ let $I = (x_1, x_2) \cap (x_3, x_4) = (x_1x_3, x_1x_4, x_2x_3, x_2x_4)$. This ideal has 4 generators, height 2, and arithmetic rank 3. The height being 2 is clear from the fact that the variety of $I$ is the union of 2 2-dimensional coordinate planes, each of which has codimension 2 (it's this last "2" that is relevant). The arithmetic rank is a 2-sided issue. First, the radical of $(x_1x_3, x_2x_4, x_1x_4 + x_2x_3)$ generate an ideal whose radical is $I$. So the arithmetic rank is no more than 3. Then an argument using "local cohomology" can be used to show it is no less than 3.

In any event, arithmetic rank is not easy to calculate. With Manoj Kummini (a postdoc here) I have a theorem that says: if you take a monomial ideal and look at its lcm-lattice (this is the collection of all possible lcm's of substes of the generators of $I$) then the longest strictly increasing chain of lcm's in the lattice is an upper bound for the arithmetic rank of $I$. (We have a recipe for writing down the right number of generators). We also have another theorem that says: if you choose a "rooting map" on this lcm-lattice then the size of an object determined by the rooting map is also an upper bound for the arithmetic rank.

What we don't know is whether this last theorem is actually better than what was known before. In order to determine this, we need to solve:

Question: Is there an LCM lattice and a rooting map on it such that the rooting complex has dimension less than the smallest rooting complex from a global ordering of the generators.

Reading material: prerpint from Manoj/Uli.

## 4. Non-reconstructible loci

Take just 4 points in $\mathbb{R}^2$ and their 6 distances. As a set (unlabelled) the distances do not change if you relabel the vertices. The same applies if you apply a Euclidean motion to the constellation. Suppose you told me the set of distances. In almost all cases, I can reconstruct from the distance data the configuration. (Of course, I can't recover *where* in the plane the constellation was, but at least their relative positions).

However, there are exceptional cases where the same set of distances arises from seriously non-similar configuration. The "almost all" business above means: the bad cases are measure zero. Actually, the bad cases even sit inside a hypersurface.

Question: for $n = 4$ (or bigger), determine the ramification locus. In the best of all worlds, this emans: find some algebraic relations between the distances such that if all relations are satisfied then reconstruction fails, while if at least one relation is not satisfied then reconstruction is unique.

Reading material: Boutin–Kemper on the arXiv..

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