HW #5

ECE 301

Prof. Mimi

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$\frac{126}{130}$

Great job!

1. $x(t) = e^{-3|t|}$

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$= \int_{-\infty}^{\infty} e^{-3|t|} e^{-j\omega t} dt$

$= \int_{-\infty}^{0} e^{t(3-j\omega)} + \int_{0}^{\infty} e^{-t(3+j\omega)}$

$= \frac{1}{3-j\omega} e^{t(3-j\omega)}\Big|_{-\infty}^{0} + \frac{1}{3+j\omega} e^{-t(3+j\omega)}\Big|_{0}^{\infty}$

$= \frac{1}{3-j\omega} + \frac{1}{3+j\omega} = \boxed{\frac{6}{9+\omega^2}}$ ✓

The table gives the solution $e^{-at}u(t) \rightarrow \frac{1}{a+j\omega}$
We essentially have the sum of the two functions:

$x(t) = e^{-3t}u(t)$

$x(t) = e^{3t}u(-t)$

Therefore the Fourier transform of our function will be the sum of the F.T. of the two functions above:

$X(\omega) = \frac{1}{3+j\omega}$

$X(\omega) = \frac{1}{3-j\omega}$ ✓

Which is exactly what is shown above.

2. $x(t) = \sin^2(\pi t + \frac{\pi}{12}) = \left[\frac{1}{2j} e^{j\pi t + j\frac{\pi}{12}} - \frac{1}{2j} e^{-j\pi t - j\frac{\pi}{12}}\right]^2$

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$x(t) = -\frac{1}{4} e^{2j\pi t + j\frac{\pi}{6}} + \frac{1}{4}e^{0} + \frac{1}{4}e^{0} - \frac{1}{4} e^{-2\pi j t - j\frac{\pi}{6}}$

$= \frac{1}{2} - \frac{1}{4} e^{2j\pi t + j\frac{\pi}{6}} - \frac{1}{4} e^{-2j\pi t - j\frac{\pi}{6}}$

$X(\omega) = \int_{-\infty}^{\infty} \left(\frac{1}{2} - \frac{1}{4} e^{2j\pi t + j\frac{\pi}{6}} - \frac{1}{4} e^{-2j\pi t - \frac{\pi}{6}}\right) e^{-j\omega t} dt$

2 cont

$$X(\omega) = \underbrace{\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-j\omega t}\,dt}_{A} - \underbrace{\tfrac{1}{4}e^{j\frac{\pi}{6}}\int_{-\infty}^{\infty} e^{2j\pi t - j\omega t}\,dt}_{B} - \underbrace{\tfrac{1}{4}e^{-j\frac{\pi}{6}}\int_{-\infty}^{\infty} e^{-2j\pi t - j\omega t}\,dt}_{C}$$

$$x_A(t) = 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_A(\omega)e^{j\omega t}\,d\omega$$
$$\rightarrow X_A(\omega) = 2\pi\delta(\omega)$$
$$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}(2\pi) = 1 \checkmark$$

$$x_B(t) = e^{2j\pi t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_B(\omega)e^{j\omega t}\,d\omega$$
$$\rightarrow X_B(\omega) = 2\pi\delta(\omega - 2\pi)$$
$$e^{2j\pi t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-2\pi)e^{j\omega t}\,d\omega$$
$$= \frac{1}{2\pi}\left(2\pi e^{j2\pi t}\right) = e^{2j\pi t} \checkmark$$

$$x_C(t) = e^{-2j\pi t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_C(\omega)e^{j\omega t}\,d\omega$$
$$\rightarrow X_C(\omega) = 2\pi\delta(\omega + 2\pi)$$
$$e^{-2j\pi t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega+2\pi)e^{j\omega t}\,d\omega$$
$$= \frac{1}{2\pi}\left(2\pi\left(e^{j(-2\pi)t}\right)\right) = e^{-2j\pi t} \checkmark$$

$$\Rightarrow X(\omega) = \tfrac{1}{2}(2\pi\delta(\omega)) - \tfrac{1}{4}e^{j\frac{\pi}{6}}(2\pi\delta(\omega-2\pi)) - \tfrac{1}{4}e^{-j\frac{\pi}{6}}(2\pi\delta(\omega+2\pi))$$

$$\boxed{= \frac{\pi}{2}\left[2\delta(\omega) - e^{j\frac{\pi}{6}}\delta(\omega-2\pi) - e^{-j\frac{\pi}{6}}\delta(\omega+2\pi)\right]} \checkmark$$

Check: $x(t) = \frac{1}{2} - \frac{1}{4}e^{j2\pi t}e^{j\frac{\pi}{6}} - \frac{1}{4}e^{j(-2\pi)t}e^{-j\frac{\pi}{6}}$

$$\tfrac{1}{2} = \tfrac{1}{2}e^{j(0)t} \rightarrow \tfrac{1}{2}(2\pi\delta(\omega-0)) = \pi\delta(\omega) \checkmark$$

$$-\tfrac{1}{4}e^{j\frac{\pi}{6}}e^{j2\pi t} \xrightarrow{\mathcal{F}} -\tfrac{1}{4}e^{j\frac{\pi}{6}}(2\pi\delta(\omega-2\pi)) = -\tfrac{\pi}{2}e^{j\frac{\pi}{6}}\delta(\omega-2\pi) \checkmark$$

$$-\tfrac{1}{4}e^{-j\frac{\pi}{6}}e^{j(-2\pi)t} \xrightarrow{\mathcal{F}} -\tfrac{1}{4}e^{-j\frac{\pi}{6}}(2\pi\delta(\omega+2\pi)) = -\tfrac{\pi}{2}e^{-j\frac{\pi}{6}}\delta(\omega+2\pi) \checkmark$$

The separate parts of $x(t)$ all transform into values in agreement with the solution $X(\omega)$ ✓

3 $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$ Parseval's Eqn.

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| |\omega|^3 \left(u(\omega+8) - u(\omega-5)\right)\right|^2 d\omega$

$= \frac{1}{2\pi}\int_{-8}^{5} \omega^6 \, d\omega$

$= \frac{1}{2\pi}\left[\frac{\omega^7}{7}\right]_{-8}^{5}$

$= \frac{1}{2\pi}\left(\frac{5^7}{7} - \frac{(-8)^7}{7}\right)$

$= \frac{1}{2\pi}\left(\frac{2175277}{7}\right) = \frac{2175277}{14\pi}$ ✓

4. $y(t) = x(-3t+2) = x\left(-3\left(t - \frac{2}{3}\right)\right)$

time shifting

$\mathcal{F}\left(x\left(t-\frac{2}{3}\right)\right) = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt$, let $t - t_0 = u$

$= \int_{-\infty}^{\infty} x(u) e^{-j\omega(u+t_0)} du = e^{-j\omega t_0}\int_{-\infty}^{\infty} x(u) e^{-j\omega t} du$

$= e^{-j\omega\frac{2}{3}} X(\omega) = X_1(\omega)$

Time Scaling

$x(at) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

$x(-3t) = \frac{1}{|-3|} X_1\left(\frac{\omega}{-3}\right)$

$\Rightarrow x(-3t+2) = \frac{e^{\frac{-2j\omega}{3}}}{3} X\left(\frac{\omega}{-3}\right)$ ✓

5 a) $h(t) = e^{-5t} u(t)$

$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$

$= \int_{0}^{\infty} e^{-t(5+j\omega)} dt$

$= -\left(\frac{1}{5+j\omega}\right) e^{-t(5+j\omega)}\Big|_0^{\infty}$

$\boxed{= \frac{1}{5+j\omega}}$

5b) $x(t) = e^{-2(t-2)} u(t-2)$

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$= \int_{-\infty}^{\infty} e^{-2(t-2)} u(t-2) e^{-j\omega t} dt$

$= \int_{2}^{\infty} e^{-2(t-2)-j\omega t} dt$

$= \int_{2}^{\infty} e^{-t(2+j\omega)+4} dt = \int_{2}^{\infty} e^{-t(2+j\omega)} e^{4} dt$

$= e^{4} \int_{2}^{\infty} e^{-t(2+j\omega)} dt$

$= e^{4} \left[-\frac{1}{(2+j\omega)} e^{-t(2+j\omega)} \Big|_{2}^{\infty} \right] = e^{4} - \left[\frac{1}{2+j\omega}\right] \left(e^{-\infty} - e^{-4-2j\omega}\right)$

$= + \frac{1}{2+j\omega} \left(e^{-2j\omega}\right)$

$Y(\omega) = H(\omega) X(\omega)$

$= \frac{1}{3+j\omega} \left(-\frac{1}{2+j\omega}\right) \left(e^{-2j\omega}\right)$

$= \left(\frac{A}{3+j\omega} - \frac{B}{2+j\omega}\right) \left(-e^{-2j\omega}\right)$ -1

$1 = A(2+j\omega) + B(3+j\omega)$

$j\omega = -2 \quad B(3-2) = 1 \Rightarrow B = 1$

$j\omega = -3 \quad A(2-3) = 1 \Rightarrow A = -1$

$Y(\omega) = \left(\frac{1}{2+j\omega} - \frac{1}{3+j\omega}\right) \left(+e^{-2j\omega}\right) = Y_1(\omega) Y_2(\omega) = y_1(t) * y_2(t)$

$y_1(t) = \left[e^{-2t} - e^{-3t}\right] u(t) \quad y_2(t) = \delta(t-2)$

by sifting property

$y(t) = \left[e^{-2t} - e^{-3t}\right] u(t) * \delta(t-2) = \boxed{\left[e^{-2(t-2)} - e^{-3(t-2)}\right] u(t-2)}$

6 a) $(j\omega)^2 Y(\omega) = 3j\omega Y(\omega) - 2Y(\omega) + X(\omega)$

$X(\omega) = (j\omega)^2 Y(\omega) - 3j\omega Y(\omega) + 2(Y(\omega))$

$= Y(\omega)((j\omega)^2 - 3j\omega + 2)$

$Y(\omega) = \dfrac{X(\omega)}{(j\omega)^2 - 3j\omega + 2}$

b) $H(\omega) = 1/(j\omega)^2 - 3j\omega + 2$

$= \dfrac{1}{(j\omega-2)(j\omega-1)}$

$= \dfrac{A}{j\omega-2} + \dfrac{B}{j\omega-1}$

$1 = (j\omega-1)A + B(j\omega-2)$

$j\omega = 2;\ (2-1)A = 1 \quad A = 1$

$j\omega = 1;\ (1-2)B = 1 \quad B = -1$

$H(\omega) = \dfrac{1}{j\omega-2} - \dfrac{1}{j\omega-1} = \dfrac{1}{1-j\omega} - \dfrac{1}{2-j\omega}$ (form looks like one in the table)

$\Downarrow$

from tables

$h(t) = (e^{2t} - e^{t})u(-t)$ $\quad$ $h(t) = e^{t}u(-t) - e^{2t}u(-t)$

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