LAB #4

First Order Linear Differential Equations

Goal: Introduction to symbolic routines in **Maple** to solve differential equations; differences in linear and nonlinear differential equations; solutions to homogeneous equations; particular solutions

Required tools: Solutions of first order linear equations; **Maple** routines: **diff, int, dsolve, evalf, subs**; Existence and Uniqueness Theorems for First Order Linear Equations; ***dfield*** .

Discussion

In this lab you will use the mathematical software program **Maple** and some of its routines to study first order linear differential equations:

$$y' + p(x)y = q(x) \qquad (*)$$

Here are a few **Maple** useful routines (use the correct syntax):

(i) *Differentiating functions* :

```
> f(x):=diff(x*exp(2*x),x);
```

Produces $\frac{d}{dx}\left\{xe^{2x}\right\}$, sets it equal to $f(x)$.

(ii) *Integrating functions* :

```
> g(x):=int(f(x),x);
```

Produces the integral of the function $f(x)$, without the "$+C$".

(iii) *Writing a differential equation* :

```
> equ1:=diff(y(x),x)+x^2*y(x)=x^2;
```

Sets the differential equation $y' + x^2 y = x^2$ as "equ1".

(iv) *Solving a differential equation* :

```
> dsolve(equ1,y(x));
```

Solves the differential equation defined by "equ1". Note that the constants will appear as $C1$, $C2$, $C3$, etc (instead of C_1, C_2, C_3).

```
> yc(x):=rhs(dsolve(equ1,y(x)));
```

Solves the differential equation and sets solution equal to $yc(x)$.

(v) *Solving a differential equation with initial condition* :

```
> dsolve({equ1,y(0)=2},y(x));
```

Produces the solution to the initial value problem $y' + x^2 y = x^2\,, \quad y(0) = 2$.

(vi) *Evaluating solutions at other values* :

> `soln:=dsolve({equ1,y(0)=2},y(x));`
Sets solution to initial value problem equal to "soln".

> `subs(x=3,rhs(soln));`
Produces the value $y(3)$ (may not be a numerical answer).

> `evalf(subs(x=3,rhs(soln)));`
Produces a numeric approximation to $y(3)$.

(vii) *Simplifying an expression*

> `simplify(%)`
Simplifies the expression immediately preceding the current expression (you may need to use this to force **Maple** to simplify the answer obtained by differentiating, integrating and/or performing substitutions).

Remark: Checking a solution

Suppose you want to check if $y = x^2$ or $y = x^3$ is a solution to the differential equation $y' - \frac{y}{x} = 2x^2$. First check $y = x^2$:

> `equ2:=diff(y(x),x)-y(x)/x=2*x^2 ;` [defines the differential equation]
> `subs(y(x)=x^2,lhs(equ2)) ;` [puts $y = x^2$ into left hand side of equation]
> `simplify(%);` [simplifies previous expression]

This last command should produce the answer "x". This is the left hand side of the differential equation above. However, the right hand side is "$2x^2$". Since these are not equal, the function $y = x^2$ is **not** a solution to the differential equation. Go back and check if $y = x^3$ is a solution. It is.

ASSIGNMENT

(1) For practice, use **Maple** routines as described above to find solutions to these initial value problems:

$$\begin{cases} (x^2+1)y' + 3xy = 6x \\ y(0) = -1 \end{cases} \qquad \begin{cases} y' = \dfrac{-2t(1+y^2)}{y} \\ y(0) = 1 \end{cases}.$$

(2) Now consider the differential equation

$$y' + x^3 y = x^3 \qquad (**)$$

(a) Let $f(x)$ be the solution to $(**)$ with $f(0) = 2$. Let $g(x)$ be the solution to $(**)$ with $g(0) =$ **seed**. Use **dsolve** to find the functions $f(x)$ and $g(x)$.

Hint: `f(x):=rhs(dsolve({equ,y(0)=2},y(x)));`

(b) Use **Maple** to substitute $f(x)$ and then $g(x)$ back into the equation $(**)$ to verify that they really do satisfy the differential equation (you may need to use the **simplify** routine in **Maple**).

(c) Let $h(x) = f(x) - g(x)$ in **Maple**, i.e.,
`> h(x):=f(x)-g(x);`
Now subsitute $h(x)$ into the left hand side (lhs) of the original differential equation $(**)$ (you may need to use **simplify**). What do you get ?

(d) The differential equation $(*)$ is said to be homogeneous if $q(x) = 0$. The above problem demonstrates an important fact: *any two solutions to* $(*)$ *differ at most by a solution to the corresponding homogeneous equation.* Prove this fact in general: namely, if $f(x)$ and $g(x)$ are solutions to $y' + p(x)y = q(x)$, then the function $h(x) = f(x) - g(x)$ is a solution to $y' + p(x)y = 0$.

(e) Use **dsolve** to find the general solution to the homogeneous differential equation $y' + x^3y = 0$. Call the solution $yc(x)$. Add the right side of your answer to the function $g(x)$ obtained in (d), i.e., let $F(x) := yc(x) + g(x)$. Substitute $F(x)$ into the left side of the differential equation. What do you get ? This demonstrates another general fact: *if we take any particular solution* $g(x)$ *to a nonhohogeneous linear equation and add it to the general solution* $yc(x)$ *of the corresponding homogeneous equation, the sum* $F(x) = yc(x) + g(x)$ *is the general solution to the original nonhomogeneous linear equation.*

(f) Use **dsolve** to find the general solution to $y' + x^3y = x^3$. Does the expression you obtain describe the same set of solutions as in (e) ? Explain.

(3) According to the basic Existence and Uniqueness Theorems, the nonlinear differential equation $y' = y^2$ is wonderful in the sense that given any initial condition $y(t_0) = y_0$ there is one and only one solution. And yet, as you saw in Lab # 3, the solution y such that $y(0) = 1$ does not exist for $t > 1$ (since $y = \frac{1}{1-t}$). Using ***dfield*** , plot several integral curves corresponding to different initial conditions. Does the place where they cease to exist seem to depend upon the initial condition? (Do not hand in this graph.)

(4) Consider the first order linear equation with initial condition:

$$(x-2)^2y' - y = x^2(x-2)^2\,, \quad y(0) = 1\,.$$

Use ***dfield*** to plot the solution (use the window $0 \le x \le 3,\ 0 \le y \le 10^4$). Plot the integral curves corresponding to $y(0) = 100$ and $y(0) = 1000$. Are both solutions unbounded near $x = 2$? How does the form of the differential equation make you expect this ? (*Hint*: Write the differential equation in normal form $y' = f(x, y)$.) Hand in your plot.

Remark: The above problems (3) and (4) illustrate a fundamental difference between linear and nonlinear differential equations. In a nonlinear differential equation it is not easy to determine if and where the solutions become unbounded or determine the interval of existence without actually solving the differential equation. However, for a linear differential equation in standard form $(*)$, the solutions can only become unbounded where either $p(x)$ or $q(x)$ are discontinuous. More precisely, the following theorem is true:

Existence & Uniqueness Theorem for 1^{st} Order Linear Equations:
If $p(x)$ and $q(x)$ are continuous on the interval (α, β) which contains x_0, then the initial value problem

$$\begin{cases} y' + p(x)y = q(x) \\ y(x_0) = y_0 \end{cases}$$

has a unique solution $y(x)$ which exists for $\alpha < x < \beta$ and any value of y_0.

(5) The above Theorem states, and problem (4) demonstrated, that the solution for a linear equation exists on any interval where the coefficients are continuous. However, you should not think that just because the function $p(x)$ or $q(x)$ become unbounded that this necessarily means the solution becomes unbounded. Use ***dfield*** to plot the solution to this initial value problem

$$y' = x^2 - \frac{1}{(x-2)^2}y\,, \quad y(0) = 1$$

with window $0 \le x \le 3$, $0 \le y \le 3$. Describe the behavior of the solution as $x \to 2$. Also, plot the solution corresponding to $y(0) = 2$. Do you see an initial condition for which the solution is not unique ? Explain why the hypotheses of the above Theorem do not hold at this point. Plot the solution corresponding to $y(3) = 1$. How does the behavior of this solution near $x = 2$ differ from the solution for $y(0) = 1$ near $x = 2$?