Tuesday, September 04, 2007
3:03 PM

Observation 1

any DT signal $x[n]$ can be written as a sum:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

In other words, any DT signal can be written as a linear combination of shifted impulses

exercise: write $x[n] = u[n]$ as a linear combination of shifted impulses:

Observation 2

The response of a DT linear system can be written as a sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h_k[n]$$

when $h_k[n]$ is the systems response to the shifted impulse $\delta[n-k]$

why? - because $x[n] = \sum x[k]\,\delta[n-k]$

by linearity: $y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot$ systems response to $\underbrace{\delta[n-k]}_{\to h_k[n]}$

Observation 3

The response of an LTI DT system can be written as an even simpler sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[n]\,h[n-k]$$

when $h[n]$ is the system's response to $\delta[n]$.

"$h[n]$ is called the unit impulse response" why? because $h_k[n]$ is a response to $\delta[n-k]$

by time invariance, $h_k[n] = h_0[n-k]$ we write $h_0[n]$ as $h[n]$

Introduce "convolution" $*$ between two functions

$$\boxed{z_1[n] * z_2[n] = \sum_{k=-\infty}^{\infty} z_1[k]\, z_2[n-k]}$$

Result – for LTI systems, $y[n] = x[n] * h[n]$

where $h[n]$ is a system's response to $\delta[n]$

How do I get $h[n]$

Ex $y[n] = 3x[n] - 2$ $\quad h[n] = 3\delta[n] - 2$

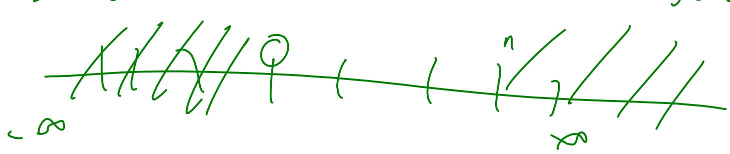
Quiz

Question: if an LTI system has unit impulse response $h[n] = u[n]$, what is the system's response due to input $x[n] = 2^n u[n]$

output is $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

$$= \sum_{k=-\infty}^{\infty} 2^k u[k]\, u[n-k] = \sum_{k=0}^{\infty} 2^k u[n-k] = \sum_{k=0}^{n} 2^k$$

but $u[n-k] = 0$ when $n-k<0$ which is $k>n$, else when $k \not> n$ else it is 1

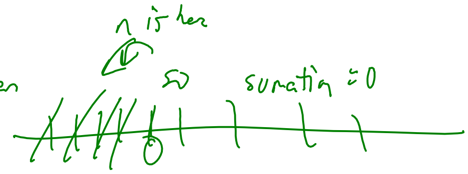


$\nearrow$ if $n \geq 0$

if $n<0$, then

n is here

so summation $= 0$



So $\left\{ \sum_{k=0}^{n} 2^k, \text{ if } n \geq 0 \right.$ $\quad (1-)^{n+1}$ ... $n+1$

so

$$y[n] = \begin{cases} \sum_{k=0}^{n} 2^k, & \text{if } n \ge 0 \\ 0, & \text{else} \end{cases} = \begin{cases} \frac{1-2^{n+1}}{1-2}, & \text{if } n \ge 0 \\ 0, & \text{else} \end{cases} = 2^{n+1}-1$$

Know:

$$\sum_{k=0}^{n} \alpha^k = \begin{cases} \frac{1-\alpha^{n+1}}{1-\alpha}, & \alpha \ne 0 \\ n+1, & \text{else} \end{cases}$$

- Geometric Series

$$\sum_{k=0}^{\infty} \alpha^k = \begin{cases} \frac{1}{1-\alpha}, & |\alpha| < 1 \\ \text{diverges}, & \text{else} \end{cases}$$

For CT:

→ convolution interval

observation 1: any CT signal can be written as an integral $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$

for any t $x(\tau)\,\delta(t-\tau) = x(t)\,\delta(t-\tau)$

because $\delta(t-\tau)$ is zero for all time except t

$$\text{so } x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,d\tau = x(t)\int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau$$

Observation 2

If a system is linear, then its response can be written as an integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h_\tau(t)\, d\tau$$

where $h_\tau(t)$ is the response to $\delta(t-\tau)$

Observation 3 If a system is LTI, then its response can be written as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

where $h(t)$ is the system's response to $\delta(t)$, $h(t)$ is where $h(t)$ is the system's response to $\delta(t)$, $h(t)$ is called "unit impulse response"

Introduce convolution $*$ between 2 CT signals

$$\boxed{z_1(t) * z_2(t) = \int_{-\infty}^{\infty} z_1(\tau) z_2(t-\tau)\, d\tau}$$

Question: The unit impulse response of an LTI system is $h(t) = u(t)$. Find the system's response to $x(t) = e^{-t}u(t)$

Answer: is, $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$$= \int_{-\infty}^{\infty} e^{-\tau} u(\tau)\, u(t-\tau)\, d\tau = \int_{0}^{\infty} e^{-\tau} u(t-\tau)\, d\tau$$

$=$, but $u(t-\tau)$ is zero when $t-\tau < 0$ ie. when $\tau > t$

else $u(t-\tau) = 1$

$$= \begin{cases} \int_0^t e^{-\tau}\, d\tau, & t \geq 0 \\ 0, & t < 0 \end{cases} = \left(\int_0^t e^{-\tau}\, d\tau\right) u(t) = \left(-e^{-\tau}\Big|_0^t\right) u(t)$$

$$= \boxed{\left(1 - e^{-t}\right) u(t)}$$

try $h(t) = u(t+7)$

$x(t) = e^{-13t}\, u(t-13)$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} 3^k u[k-1]\, u[n-k]$$

$$= \sum_{1}^{\infty} 3^k u[n-k] = \begin{cases} \sum_{1}^{n} 3^k & \text{if } n \geq 1 \\ 0 & \text{, else} \end{cases}$$

$$= \text{let } r = k-1 \quad \left(\sum_{0}^{n-1} 3^{r+1}\right) u[n-1] = \left(3 \sum_{0}^{n-1} 3^r\right) u[n-1] = \left(3 \cdot \frac{1-3^n}{1-3}\right) u[n-1]$$

$= \frac{3}{2}(3^n - 1)\ u[n-1]$

$h[n] = u[n]$

$x[n] = 2^n u[n]$

$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

$= \sum_{k=-\infty}^{\infty} 2^k u[k]\ u[n-k] = \sum_{k=0}^{\infty} 2^k u[n-k]$

$= \begin{cases} \sum_{k=0}^{n} 2^k & , \text{if } n \geq 0 \\ 0 & , \text{else} \end{cases} = \left(\frac{1-2^{n+1}}{1-2}\right) u[n]$